Definition

There are various ways to define the determinant of a square matrix *A*, i.e. one with the same number of rows and columns. Perhaps the most natural way is expressed in terms of the columns of the matrix. If we write an $n \times n$ matrix in terms of its column vectors

$$A = \begin{bmatrix} a_1, & a_2, & \dots, & a_n \end{bmatrix}$$

where the a_j are vectors of size *n*, then the determinant of *A* is defined so that

$$\begin{aligned}
&\det \begin{bmatrix} a_1, & \dots, & ba_j + cv, & \dots, a_n \end{bmatrix} = b\det(A) + c\det \begin{bmatrix} a_1, & \dots, & v, & \dots, a_n \end{bmatrix} \\
&\det \begin{bmatrix} a_1, & \dots, & a_j, & a_{j+1}, & \dots, a_n \end{bmatrix} = -\det \begin{bmatrix} a_1, & \dots, & a_{j+1}, & a_j, & \dots, a_n \end{bmatrix} \\
&\det(I) = 1
\end{aligned}$$

where *b* and *c* are scalars, *v* is any vector of size *n* and *I* is the identity matrix of size *n*. These equations say that the determinant is a linear function of each column, that interchanging adjacent columns reverses the sign of the determinant, and that the determinant of the identity matrix is 1. These properties mean that the determinant is an alternating multilinear function of the columns that maps the identity matrix to the underlying unit scalar. These suffice to uniquely calculate the determinant of any square matrix. Provided the underlying scalars form a field (more generally, a commutative ring with unity), the definition below shows that such a function exists, and it can be shown to be unique.[2]

Equivalently, the determinant can be expressed as a sum of products of entries of the matrix where each product has *n* terms and the coefficient of each product is −1 or 1 or 0 according to a given rule: it is a polynomial expression of the matrix entries. This expression grows rapidly with the size of the matrix (an $n \times n$ matrix contributes $n!$ terms), so it will first be given explicitly for the case of 2×2 matrices and 3×3 matrices, followed by the rule for arbitrary size matrices, which subsumes these two cases.

Assume *A* is a square matrix with *n* rows and *n* columns, so that it can be written as

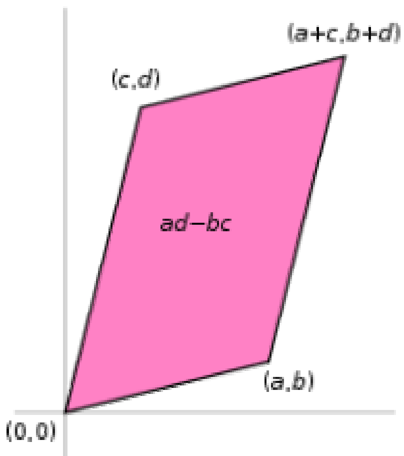
$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix}.$$

The entries can be numbers or expressions (as happens when the determinant is used to define a characteristic polynomial); the definition of the determinant depends only on the fact that they can be added and multiplied together in a commutative manner.

The determinant of A is denoted as det(A), or it can be denoted directly in terms of the matrix entries by writing enclosing bars instead of brackets:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}.$$

2 × 2 matrices



The area of the parallelogram is the absolute value of the determinant of the matrix formed by the vectors representing the parallelogram's sides.

The determinant of a 2 × 2 matrix is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

If the matrix entries are real numbers, the matrix A can be used to represent two linear maps: one that maps the standard basis vectors to the rows of A, and one that maps them to the columns of A. In either case, the images of the basis vectors form a parallelogram that represents the image of the unit square under the mapping. The parallelogram defined by the rows of the above matrix is the one with vertices at (0, 0), (a, b), ($a + c$, $b + d$), and (c, d), as shown in the accompanying diagram. The absolute value of $ad - bc$ is the area of the parallelogram, and thus represents the scale factor by which areas are transformed by A. (The parallelogram formed by the columns of A is in general a

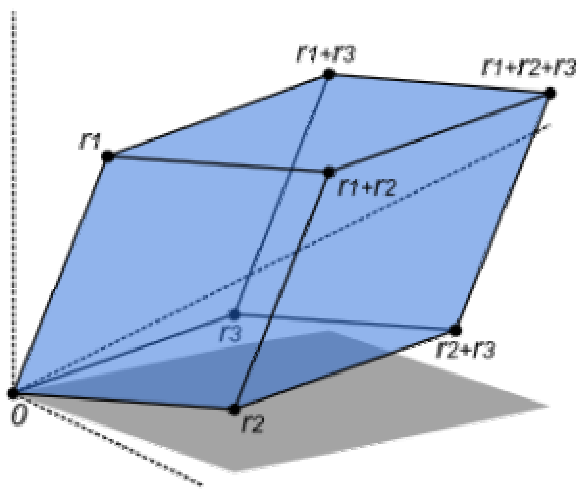
different parallelogram, but since the determinant is symmetric with respect to rows and columns, the area will be the same.)

The absolute value of the determinant together with the sign becomes the *oriented area* of the parallelogram. The oriented area is the same as the usual area, except that it is negative when the angle from the first to the second vector defining the parallelogram turns in a clockwise direction (which is opposite to the direction one would get for the identity matrix).

Thus the determinant gives the scaling factor and the orientation induced by the mapping represented by A. When the determinant is equal to one, the linear mapping defined by the matrix is equi-areal and orientation-preserving.

The object known as the *bivector* is related to these ideas. In 2D, it can be interpreted as an *oriented plane segment* formed by imagining two vectors each with origin (0, 0), and coordinates (a, b) and (c, d). The bivector magnitude (denoted $(a, b) \wedge (c, d)$) is the *signed area*, which is also the determinant $ad - bc$.[3]

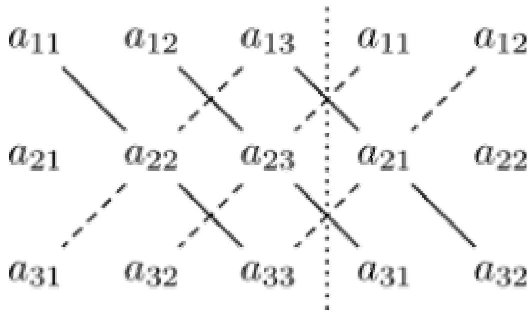
3 × 3 matrices



The volume of this parallelepiped is the absolute value of the determinant of the matrix formed by the rows constructed from the vectors r1, r2, and r3.

The determinant of a 3 × 3 matrix is defined by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$
$$= aei + bfg + cdh - ceg - bdi - afh.$$



Sarrus' rule: The determinant of the three columns on the left is the sum of the products along the solid diagonals minus the sum of the products along the dashed diagonals

The rule of Sarrus is a mnemonic for the 3 × 3 matrix determinant: the sum of the products of three diagonal north-west to south-east lines of matrix elements, minus the sum of the products of three diagonal south-west to north-east lines of elements, when the copies of the first two columns of the matrix are written beside it as in the illustration. This scheme for calculating the determinant of a 3 × 3 matrix does not carry over into higher dimensions.

n × *n* matrices

The determinant of a matrix of arbitrary size can be defined by the Leibniz formula or the Laplace formula.

The Leibniz formula for the determinant of an $n \times n$ matrix A is

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}.$$

Here the sum is computed over all permutations σ of the set $\{1, 2, ..., n\}$. A permutation is a function that reorders this set of integers. The value in the ith position after the reordering σ is denoted σ_i. For example, for $n = 3$, the original sequence 1, 2, 3 might be reordered to $\sigma = [2, 3, 1]$, with $\sigma_1 = 2$, $\sigma_2 = 3$, and $\sigma_3 = 1$. The set of all such permutations (also known as the symmetric group on n elements) is denoted S_n. For each permutation

σ, sgn(σ) denotes the signature of σ, a value that is +1 whenever the reordering given by σ can be achieved by successively interchanging two entries an even number of times, and −1 whenever it can be achieved by an odd number of such interchanges.

In any of the $n!$ summands, the term

$$\prod_{i=1}^{n} a_{i,\sigma_i}$$

is notation for the product of the entries at positions (i, σ_i), where i ranges from 1 to n:

$$a_{1,\sigma_1} \cdot a_{2,\sigma_2} \cdots a_{n,\sigma_n}.$$

For example, the determinant of a 3 × 3 matrix A (n = 3) is

$$\begin{aligned}\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i} &= \operatorname{sgn}([1,2,3]) \prod_{i=1}^{n} a_{i,[1,2,3]_i} + \operatorname{sgn}([1,3,2]) \prod_{i=1}^{n} a_{i,[1,3,2]_i} + \operatorname{sgn}([2,1,3]) \prod_{i=1}^{n} a_{i,[2,1,3]_i} \\ &+ \operatorname{sgn}([2,3,1]) \prod_{i=1}^{n} a_{i,[2,3,1]_i} + \operatorname{sgn}([3,1,2]) \prod_{i=1}^{n} a_{i,[3,1,2]_i} + \operatorname{sgn}([3,2,1]) \prod_{i=1}^{n} a_{i,[3,2,1]_i} \\ &= \prod_{i=1}^{n} a_{i,[1,2,3]_i} - \prod_{i=1}^{n} a_{i,[1,3,2]_i} - \prod_{i=1}^{n} a_{i,[2,1,3]_i} + \prod_{i=1}^{n} a_{i,[2,3,1]_i} + \prod_{i=1}^{n} a_{i,[3,1,2]_i} - \prod_{i=1}^{n} a_{i,[3,2,1]_i} \\ &= a_{1,1}a_{2,2}a_{3,3} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} \\ &\quad + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1}.\end{aligned}$$

Properties of the determinant

The determinant has many properties. Some basic properties of determinants are:

1. $\det(I_n) = 1$ where I_n is the n × n identity matrix.
2. $\det(A^T) = \det(A).$
3. $\det(A^{-1}) = \dfrac{1}{\det(A)} = \det(A)^{-1}.$
4. For square matrices A and B of equal size,

   $$\det(AB) = \det(A)\det(B).$$

5. $\det(cA) = c^n \det(A)$ for an $n \times n$ matrix.
6. If A is a triangular matrix, i.e. $a_{i,j} = 0$ whenever $i > j$ or, alternatively, whenever $i < j$, then its determinant equals the product of the diagonal entries:

$$\det(A) = a_{1,1}a_{2,2}\cdots a_{n,n} = \prod_{i=1}^{n} a_{i,i}.$$

This can be deduced from some of the properties below, but it follows most easily directly from the Leibniz formula (or from the Laplace expansion), in which the identity permutation is the only one that gives a non-zero contribution.

A number of additional properties relate to the effects on the determinant of changing particular rows or columns:

7. Viewing an $n \times n$ matrix as being composed of n columns, the determinant is an n-linear function. This means that if one column of a matrix A is written as a sum $v + w$ of two column vectors, and all other columns are left unchanged, then the determinant of A is the sum of the determinants of the matrices obtained from A by replacing the column by v and then by w (and a similar relation holds when writing a column as a scalar multiple of a column vector).
8. If in a matrix, any row or column is 0, then the determinant of that particular matrix is 0.
9. This n-linear function is an alternating form. This means that whenever two columns of a matrix are identical, or more generally some column can be expressed as a linear combination of the other columns (i.e. the columns of the matrix form a linearly dependent set), its determinant is 0.

Properties 1, 7 and 9 — which all follow from the Leibniz formula — completely characterize the determinant; in other words the determinant is the unique function from $n \times n$ matrices to scalars that is n-linear alternating in the columns, and takes the value 1 for the identity matrix (this characterization holds even if scalars are taken in any given commutative ring). To see this it suffices to expand the determinant by multi-linearity in the columns into a (huge) linear combination of determinants of matrices in which each column is a standard basis vector. These determinants are either 0 (by property 8) or else ± 1 (by properties 1 and 11 below), so the linear combination gives the expression above in terms of the Levi-Civita symbol. While less technical in appearance, this characterization cannot entirely replace the Leibniz formula in defining the determinant, since without it the existence of an appropriate function is not clear. For matrices over non-commutative rings, properties 7 and 8 are incompatible for $n \geq 2$,[4] so there is no good definition of the determinant in this setting.

Property 2 above implies that properties for columns have their counterparts in terms of rows:

10. Viewing an $n \times n$ matrix as being composed of n rows, the determinant is an n-linear function.
11. This n-linear function is an alternating form: whenever two rows of a matrix are identical, its determinant is 0.
12. Interchanging two columns or rows of a matrix multiplies its determinant by −1. This follows from properties 7 and 9 (it is a general property of multilinear alternating maps). More generally, any permutation of the rows or columns multiplies the determinant by the sign of the permutation. (by permutation, it is meant viewing each row as a vector $\mathbf{R}_i$ (equivalently each column as $\mathbf{C}_i$) and reordering the rows (or columns) by interchange of $\mathbf{R}_j$ and $\mathbf{R}_k$ (or $\mathbf{C}_j$ and $\mathbf{C}_k$), where j,k are two indices chosen from 1 to n for an $n \times n$ square matrix.
13. Adding a scalar multiple of one column to *another* column does not change the value of the determinant. This is a consequence of properties 7 and 8: by property 7 the determinant changes by a multiple of the determinant of a matrix with two equal columns, which determinant is 0 by property 8. Similarly, adding a scalar multiple of one row to another row leaves the determinant unchanged.

Property 5 says that the determinant on n x n matrices is homogeneous of degree n. These properties can be used to facilitate the computation of determinants by simplifying the matrix to the point where the determinant can be determined immediately. Specifically, for matrices with coefficients in a field, properties 11 and 12 can be used to transform any matrix into a triangular matrix, whose determinant is given by property 6; this is essentially the method of Gaussian elimination.

For example, the determinant of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix}$$

can be computed using the following matrices:

$$B = \begin{bmatrix} -2 & 2 & -3 \\ 0 & 0 & 4.5 \\ 2 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 2 & -3 \\ 0 & 0 & 4.5 \\ 0 & 2 & -4 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 2 & -3 \\ 0 & 2 & -4 \\ 0 & 0 & 4.5 \end{bmatrix}.$$

Here, B is obtained from A by adding $-1/2\times$the first row to the second, so that $\det(A) = \det(B)$. C is obtained from B by adding the first to the third row, so that $\det(C) = \det(B)$. Finally, D is obtained from C by exchanging the second and third row, so that $\det(D) = -\det(C)$. The determinant of the (upper) triangular matrix D is the product of its entries on the main diagonal: $(-2) \cdot 2 \cdot 4.5 = -18$. Therefore, $\det(A) = -\det(D) = +18$.

Multiplicativity and matrix groups

The determinant of a matrix product of square matrices equals the product of their determinants:

$$\det(AB) = \det(A)\det(B).$$

Thus the determinant is a *multiplicative map*. This property is a consequence of the characterization given above of the determinant as the unique n-linear alternating function of the columns with value 1 on the identity matrix, since the function $M_n(K) \to K$ that maps $M \mapsto \det(AM)$ can easily be seen to be n-linear and alternating in the columns of M, and takes the value $\det(A)$ at the identity. The formula can be generalized to (square) products of rectangular matrices, giving the Cauchy–Binet formula, which also provides an independent proof of the multiplicative property.

The determinant $\det(A)$ of a matrix A is non-zero if and only if A is invertible or, yet another equivalent statement, if its rank equals the size of the matrix. If so, the determinant of the inverse matrix is given by

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

In particular, products and inverses of matrices with determinant one still have this property. Thus, the set of such matrices (of fixed size n) form a group known as the special linear group. More generally, the word "special" indicates the subgroup of another matrix group of matrices of determinant one. Examples include the special orthogonal group (which if n is 2 or 3 consists of all rotation matrices), and the special unitary group.