Observation 1: A left R-module A is isomorphic to a coproduct $\coprod_{i\,\epsilon\Omega} Bi$ of a family of left R-modules $\{B_i \mid i\ \epsilon\Omega\}$ if and only if there exists a family $\{A_i \mid i\ \epsilon\Omega\}$of submodules of A such that $A_i \cong B_i$ for each $i\ \epsilon\Omega$ and such that for each element a of A there is a unique family $\{a_i \mid i\ \epsilon\Omega\}$of elements of A satisfying the following conditions:

(a) $a_i\ \epsilon\ A_i$ for all $i\ \epsilon\Omega$;

(b) $a_i = 0$ for all but finitely-many $i\ \epsilon\Omega$;

(c) $a = \sum_{i\,\epsilon\Omega} a_i$

Verification: Set $B = \coprod_{i\,\epsilon\Omega} Bi$. Suppose that A contains a family $\{A_i \mid i\ \epsilon\Omega\}$ of submodules satisfying the given conditions. For each $h\epsilon\Omega$, let $\lambda_h : B_h \rightarrow B$ be the injection of B_h into the coproduct of the B_i and let $\alpha_h : B_h \rightarrow A$ be the map which is the composite of an isomorphism $B_h \rightarrow A$ (which exists by our supposition) and the inclusion map $A_h \rightarrow A$. Define the map α: $B \rightarrow A$ to be $\sum_{i\,\epsilon\Omega} \alpha_i$.

We will show that α is an isomorphism. Let $f\ \epsilon\ \ker(\alpha)$. Then $0 = \sum_{i\,\epsilon\Omega} f(i)\alpha_i$. Let $\{z_i \mid i\ \epsilon\Omega\}$be the family of elements of A defined by $z_i = 0$ for all $i\ \epsilon\Omega$. Then the two families $\{ f(i)\alpha_i \mid i\ \epsilon\Omega\}$and $\{z_i \mid i\ \epsilon\Omega\}$of elements of A satisfy conditions (a), (b), and (c) of the observation for the case $a = 0$ and so, by the uniqueness assumption, must be equal. Thus $f(i)\alpha_i = 0$ for all $i\ \epsilon\Omega$. Since each α_i is an R-monomorphism, this implies that $f(i) = 0$ for all $i\ \epsilon\Omega$ and so α must be an R-monomorphism. To show that α is an R-epimorphism, consider an arbitrary element a of A and construct an element f of B in the following manner. By our assumption, we can write $a = \sum_{i\,\epsilon\Omega} a_i$ where each a_i belongs to the image of α_i. Therefore for each $i\ \epsilon\Omega$ there exists an element b_i of B_i such that $a = \sum_{i\,\epsilon\Omega} b_i\alpha_i$. Moreover, all but finitely-many of these b_i are equal to 0. Now let f be the function in B defined by $f(i) = b_i$ for all $i\ \epsilon\Omega$. Then $f\alpha = \sum_{i\,\epsilon\Omega} b_i\alpha_i = a$, showing that $a\ \epsilon \mathrm{im}(\alpha)$ and hence establishing that α is an R-epimorphism and so an R-isomorphism.

Now suppose that there exists an R-isomorphism α: $B \rightarrow A$. For each $i\ \epsilon\Omega$, set $A_i = B_i\,\alpha$. This is a submodule of A and it is left as an exercise to show that the family $\{A_i \mid i\ \epsilon\Omega\}$of submodules of A satisfies the conditions of the observation. □

If $\{A_i \mid i\ \epsilon\Omega\}$is a family of submodules of a left R-module A which satisfies the conditions given in Observation 1, then A is said to be the direct sum of the A_i, and we

write A $=\bigoplus_{i\,\epsilon\Omega} A_i$. It is necessary and important to distinguish between the notion of the direct sum of a family of submodules of a given module and the coproduct of an arbitrary family of modules. These concepts, however, are closely related; as we have seen, if a left R-module is isomorphic to a coproduct of a family of left R-modules then it is a direct sum of submodules isomorphic to members of that family. Therefore, many authors use the same notation, "$\bigoplus$", for both direct sums and coproducts. No harm comes of this as a rule, so long as one makes clear what is being intended at each step.

We will often be interested in showing that a module is the direct sum of two of its submodules. In that case, a slight variation of Observiation 1 yields the following result.

Observation 2: Let B and C be submodules of a left R-module A. Then $A = B \bigoplus C$ if and only if B ∩ C = (0) and for each a in A there are elements b in B and c in C for which $a = b + c$.

8. Splitting maps and summands

The projection π_A and the injection λ_A associated with a coproduct $A\coprod B$ of left R-modules have the following two properties: the composite map $\lambda_A\ \pi_A$ is the identity map on A and $A\bigoplus B$ is the direct sum of im(λ_A) and ker(π_A) (= im(λ_B)). A comparison of these two properties of π_A and λ_A motivates the following observation.

Observation 1: Let α: $A \to B$ and β: $B \to A$ be homomorphisms between left R-modules A and B for which $\alpha\beta$ is the identity map on A. Then B = im(α) $\bigoplus$ ker(β).

Verification: Let $x\ \epsilon$ im(α) ∩ker(β). Then $x = a\ \alpha$ for some $a\ \epsilon\ A$ and also $x\,\beta = 0$. Consequently, $a = a\ \alpha\beta$ = xβ = 0 and $x = a\ \alpha$ =0 = 0. Thus im(α) ∩ker(β)= (0). Let b ϵB. Then $b\ \beta\alpha\ \epsilon$ im(α) and $b = b\ \beta\alpha + [b - b\ \beta\alpha]$. To complete the verification, we need only show that $b - b\ \beta\alpha$ is in ker(β), which is indeed the case since $(b - b\ \beta\alpha)\,\beta$ $=b\ \beta - b\ \beta\alpha\ \beta = b\ \beta - b\ \beta = 0$. □

When maps α: $A \to B$ and β: $B \to A$ have the property that $\alpha\ \beta$ is the identity map on A, we say that each of the maps is a splitting map for the other. As Observation 1 shows, splitting maps are valuable tools for splitting modules into direct sums of submodules. Note that, when $\alpha\ \beta$ is the identity map on A it follows that α is an R-monomorphism and β is an R-epimorphism. The maps π_A and λ_A mentioned above are splitting maps of each other.

A submodule B of a left R-module A is a direct summand of A if there exists a submodule C of A for which $A = B \oplus C$: When such a C exists, it is said to be a complementary summand of B. Any left R-module A can be written as $A \oplus (0)$, and consequently A and (0) are complementary summands of each other. Observation 1 provides the following useful strategies for verifying that certain submodules are direct summands. If α: $A \rightarrow B$ is an R-monomorphism and we wish to show that im(α) is a direct summand of B, it is sufficient to find a map β: $B \rightarrow A$ for which $a\, \alpha\, \beta = a$ for all a in A. If α: $A \rightarrow B$ is an R-epimorphism and we wish to show that ker(α) is a direct summand of A, then it is sufficient to find a map β: $B \rightarrow A$ for which $b\, \beta\alpha = b$ for all b in B. Finally, if A is a submodule of B and we wish to show that it is a direct summand, it is sufficient to find a map β: $B \rightarrow A$ for which $a\, \beta = a$ for all a in A.

A direct summand B of a left R-module A may have many distinct complementary summands, as Exercises 1 and 2 show. There is a one-to-one correspondence between the set of all complementary summands of B and the set of splitting maps for the inclusion map of B into A. The complementary summands for the summand im(λ_B) of the coproduct $A \coprod B$ are precisely the graphs of the homomorphisms of A into B.

EXERCISES

1: Let A be a noncyclic $\mathbb{Z}$-module of order 4. Show that A is a direct sum of any two of its three cyclic submodules of order 2.

2: Let R be the field $\mathbb{Z}$ /(2) of integers modulo 2: Show that each submodule of order 2n of each R-module of order 2n+1 is a direct summand having precisely 2n distinct complementary summands.

3: Let B be a direct summand of a left R-module A and let i: $B \rightarrow A$ be the inclusion map. Show that the function associating each splitting map β: $B \rightarrow A$ of i with ker(β) is a one-to-one correspondence between the set of splitting maps of α and the set of all complementary summands of B in A.

4: Let B be a direct summand of a left R-module A. Show that the complementary summands of the submodule im(λ_B) of the left R-module $A \coprod B$ are precisely the graphs of the homomorphisms of A into B; as claimed above.

5: Let A be the left $\mathbb{Z}$ -module $\mathbb{Z}$ /(2) $\coprod \mathbb{Z}$ / (4). Let B ={(0, 0), (1, 0)} and let C = {(0, 0), (1, 2)}. Show that both B and C are direct summands of A, but that $B \oplus C$ is not.

6: Let B be a nonzero left R-module and let A be a submodule of B: Assume that C is a maximal element of the set of all submodules C of B satisfying $A \cap C$ = (0). (Note that this set is nonempty, since it surely contains (0).) Is it necessarily true that $B = A \oplus C$?