Chapter Three: Topics in Differentiation

Implicit Differentiation:

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

Example: Find dy/dx if $y^2 = x^2 + \sin xy$.

Sol: We differentiate the equation implicity.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy)$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$$

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + y\cos xy$$

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$

H.W. Find y' if $yx^2 + e^y = x$.

H.W. Find $\frac{dy}{dx}$ if $x^3 + y^3 = 3xy$.

Derivatives of Higher Order

Implicit differentiation can also be used to find higher derivatives.

Example: Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Sol.: To start, we differentiate both sides of the equation with respect to x to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8) \Rightarrow 6x^2 - 6yy' = 0$$

$$\Rightarrow 6yy' = 6x^2 \Rightarrow y' = \frac{x^2}{y}, \text{ where } y \neq 0$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{y(2x) - x^2y'}{y^2} = \frac{2xy}{y^2} - \frac{x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2y'}{y^2}$$

Finally, we substitute $y' = \frac{x^2}{y}$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \text{ where } y \neq 0$$

H.W. Find d^2y/dx^2 if $x^2 + y^2 = 25$.

H.W. Find y'' if $x^2 + y^4 = 10$.

L'Hopital's Rule:

Theorem: Suppose that f and g are differentiable functions on an open interval containing $x=a$, except possibly at $x=a$, and that

$$\lim_{x\to a} f(x)=0 \quad \text{and} \quad \lim_{x\to a} g(x)=0$$

If $\lim_{x\to a}[f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x\to a^-$, $x\to a^+$, $x\to\infty^-$ or as $x\to\infty^+$.

Example: Find the limit $\lim_{x\to 2}\frac{x^2-4}{x-2}$ using L'Hopital Rule.

Sol.: $\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{0}{0}$.

Appling L'Hopital's Rule yields

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{2x}{1}=4$$

Example: Evalute the limit $\lim_{x\to 3}\frac{x^2+x-12}{x^2-9}$ using L'Hopital's Rule.

Sol.: $\lim_{x\to 3}\frac{x^2+x-12}{x^2-9}=\frac{0}{0}$. Appling L'Hopital's Rule yields

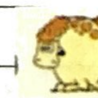
$$\lim_{x\to 3}\frac{x^2+x-12}{x^2-9}=\lim_{x\to 3}\frac{2x+1}{2x}=\frac{6+1}{6}=\frac{7}{6}$$

Indeterminate Forms:

The following expressions are all called indeterminate forms.

$0/0$, ∞/∞ , $0.\infty$, ∞^0 , 0^0 , 1^∞ , $\infty - \infty$.

The typical way to resolve $0/0$ or ∞/∞ is to use L'Hopital's Rule. And



to resolve $0.\infty$ is to convert it to $0/0$ or ∞/∞,

In the case 1^∞, the usual dack is to take the logarithm of the expression in quastion.

For both 0^0 and ∞^0, the usual technique is the same as for 1^∞.

Finally, the typical approach to resolving $\infty - \infty$ is to re-express the expression as a fraction, by finding some common denominator.

Example: Evaluate $\lim_{x \to \infty} (x . \sin \frac{2}{x})$

Sol.: $\lim_{x \to \infty} (x . \sin \frac{2}{x}) = 0 . \infty$.

Now we try to convert it to $0/0$

$\lim_{x \to \infty} \left(\frac{\sin \frac{2}{x}}{1/x} \right) = \frac{0}{0}$. Appling L'Hopital's Rule yields

$$\lim_{x \to \infty} \left(\frac{\sin \frac{2}{x}}{1/x} \right) = \lim_{x \to \infty} \left(\frac{\cos(2/x)(-2/x^2)}{-1/x^2} \right) = \lim_{x \to \infty} \left(2 \cos \frac{2}{x} \right)$$

Example: Find $\lim\limits_{h\to 0}(1+2h)^{1/h}$

Sol.: $\lim\limits_{h\to 0}(1+2h)^{1/h} = 1^{\infty}$.

$$\lim_{h\to 0}(1+2h)^{1/h} = \lim_{h\to 0} e^{\ln(1+2h)^{1/h}} = e^{\lim\limits_{h\to 0}\ln(1+2h)^{1/h}}$$

$$\lim_{h\to 0}\ln(1+2h)^{1/h} = \lim_{h\to 0}\frac{1}{h}\ln(1+2h) \quad [\text{has form } 0.\infty]$$

$$= \lim_{h\to 0}\frac{\ln(1+2h)}{h} \quad \left[\text{has form } \tfrac{0}{0}\right]$$

$$= \lim_{h\to 0}\frac{2/(1+2h)}{1} \quad [\text{L'Hopital's Rule}]$$

$$= 2$$

Then $\lim\limits_{h\to 0}(1+2h)^{1/h} = e^{2}$.

Example: Find $\lim\limits_{x\to\infty} x^{1/x}$.

Sol.: $\lim\limits_{x\to\infty} x^{1/x} = \infty^{0}$.

$$\lim_{x\to\infty} x^{1/x} = e^{\lim\limits_{x\to\infty}\ln x^{1/x}}$$

$$\lim_{x\to\infty}\ln x^{1/x} = \lim_{x\to\infty}\frac{1}{x}\ln x = \lim_{x\to\infty}\frac{\ln x}{x} \quad \left[\text{form } \tfrac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty}\frac{1/x}{1} = 0 \Rightarrow \lim_{x\to\infty} x^{1/x} = e^{0} = 1.$$

Example: Evaluate $\lim\limits_{x\to 0^+} x^x$.

Sol.: $\lim\limits_{x\to 0^+} x^x = 0^0$.

$$\lim_{x\to 0^+} x^x = e^{\lim\limits_{x\to 0^+} \ln x^x}.$$

$$\lim_{x\to 0^+} \ln x^x = \lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} \quad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

$$\lim_{x\to 0^+} x^x = e^0 = 1$$

Example: Evaluate $\lim\limits_{x\to\infty} (\sqrt{x^2+x} - x)$.

Sol.: $\lim\limits_{x\to\infty} (\sqrt{x^2+x} - x) = \infty - \infty$.

$$\sqrt{x^2+x} - x = \left(\sqrt{x^2+x} - x\right) \cdot \frac{\sqrt{x^2+x} + x}{\sqrt{x^2+x} + x} = \frac{(x^2+x) - x^2}{\sqrt{x^2+x} + x}$$

$$= \frac{x}{\sqrt{x^2+x} + x} \Longrightarrow \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x} + x} = \frac{\infty}{\infty}.$$

$$\frac{x}{\sqrt{x^2+x} + x} = \frac{x}{\sqrt{x^2+x} + x} \cdot \frac{1/x}{1/x} = \frac{1}{1/x\sqrt{x^2+x+x}} = \frac{1}{\sqrt{1+\frac{1}{x}} + 1}$$

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+x} + x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}} + 1} = \frac{1}{2}$$

Then $\lim\limits_{x\to\infty} \sqrt{x^2+x} - x = \frac{1}{2}$.

Definition: Velocity (instantaneous velocity) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Definition: Acceleration is the deriviative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's accleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

Example: A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec. It reaches a height of $s = 160t - 16t^2$ ft after t sec. what is its velocity and acceleration at any time t?

Sol.: At any time t during the rock's motion, its velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec.}$$

At any time during its flight following the explosion, the rock's acceleration is a constant

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

Definition: If $f(x)$ is a differentiable real function at $x = a$, then the approximating function

$$t(x) = f(a) + f'(a)(x-a)$$

is called the linearization of f at a

The approximation $f(x) \approx t(x)$ of f by t is the standard linear approximation (Local Linear approximation) of f at a.

The point $x = a$ is the center of the approximation.

Example: Find the linearization of $f(x) = x^2$ at $x = 1$:

Sol.: $f'(x) = 2x$.

For $x = 1$ we have $f(1) = 1$, $f'(1) = 2$.

The linearization is $t(x) = 1 + 2(x-1) = 2x - 1$

Example: Find the Local linear approximation of $f(x) = \sqrt{x}$ at $a = 1$.

Sol.: $f'(x) = 1/(2\sqrt{x})$

$$\sqrt{x} \approx \sqrt{a} + \frac{1}{2\sqrt{a}}(x-a)$$

Thus, the local linear approximation at $a = 1$ is

$$\sqrt{x} \approx 1 + \frac{1}{2}(x-1).$$