Limiting Distribution

Theorem 2. *Let the random variable Y_n have the distribution function $F_n(y)$ and the moment-generating function $M(t; n)$ that exists for $-h < t < h$ for all n. If there exists a distribution function $F(y)$, with corresponding moment-generating function $M(t)$, defined for $|t| \le h_1 < h$, such that $\lim_{n\to\infty} M(t; n) = M(t)$, then Y_n has a limiting distribution with distribution function $F(y)$.*

المبرهنة: اذا كان لدينا متغير عشوائي Y_n بدالة مولدة للعزوم $M(t;n)$ حيث ان
$-h < t < h$ لكل n ، ودالة توزيعه هي $F_n(y)$ فأذا وجدت دالة توزيع $F(y)$
بدالة مولدة للعزوم $M(t)$ حيث ان $|t| \le h_1 < h$ بحيث ان
$\lim_{n\to\infty} M(t;n) = M(t)$ فأن Y_n لها توزيع غاية بدالة توزيع.

فأذا كان بالإمكان الحصول على الصيغة التالية للدالة المولدة للعزوم

$$\lim_{n\to\infty} M(t;n) = \lim_{n\to\infty}\left[1+\frac{b}{n}+\frac{\psi(n)}{n}\right]^{cn} = \lim_{n\to\infty}\left[1+\frac{b}{n}\right]^{cn} = e^{bc}$$

حيث ان b, c لا تعتمد على n وان $\lim_{n\to\infty}\psi(n) = 0$ وبالتالي يكون

$$\lim_{n\to\infty} M(t;n) = \lim_{n\to\infty}\left[1+\frac{b}{n}\right]^{cn} = e^{bc}$$

For example,

$$\lim_{n\to\infty}\left(1-\frac{t^2}{n}+\frac{t^3}{n^{3/2}}\right)^{-n/2} = \lim_{n\to\infty}\left(1-\frac{t^2}{n}+\frac{t^3/\sqrt{n}}{n}\right)^{-n/2}.$$

Here $b = -t^2$, $c = -\frac{1}{2}$, and $\psi(n) = t^3/\sqrt{n}$. Accordingly, for every fixed value of t, the limit is $e^{t^2/2}$.

Example 1. Let Y_n have a distribution that is $b(n, p)$. Suppose that the mean $\mu = np$ is the same for every n; that is, $p = \mu/n$, where μ is a constant.

We shall find the limiting distribution of the binomial distribution, when $p = \mu/n$, by finding the limit of $M(t; n)$. Now

$$M(t; n) = E(e^{tY_n}) = [(1-p) + pe^t]^n = \left[1 + \frac{\mu(e^t - 1)}{n}\right]^n$$

for all real values of t. Hence we have

$$\lim_{n\to\infty} M(t; n) = e^{\mu(e^t-1)}$$

$$\lim_{n\to\infty} M(t;n) = e^{\mu(e^t-1)} = \mathbf{M.G.F.\ of\ P(\mu)}$$

وبالتالي اذا اصبح لمتغير عشوائي توزيع غاية فيمكن استخدام هذا التوزيع كتقريب للتوزيع الأساسي عندما تكون n كبيرة وهذه فائدة ، فمثلا يكون من السهل تكوين جداول لتوزيع بواسون بمعلمة واحدة مثلا المتغير y بتوزيع بواسون بالمعلمتين n=50, p=1/25 فأن

$$\Pr(Y \le 1) = (\tfrac{24}{25})^{50} + 50(\tfrac{1}{25})(\tfrac{24}{25})^{49} = 0.400,$$

approximately. Since $\mu = np = 2$, the Poisson approximation to this probability is

$$e^{-2} + 2e^{-2} = 0.406.$$

Example 2. Let Z_n be $\chi^2(n)$. Then the moment-generating function of Z_n is $(1 - 2t)^{-n/2}$, $t < \frac{1}{2}$. The mean and the variance of Z_n are, respectively, n and $2n$. The limiting distribution of the random variable $Y_n = (Z_n - n)/\sqrt{2n}$ will be investigated. Now the moment-generating function of Y_n is

$$\begin{aligned} M(t; n) &= E\left\{\exp\left[t\left(\frac{Z_n - n}{\sqrt{2n}}\right)\right]\right\} \\ &= e^{-tn/\sqrt{2n}} E(e^{tZ_n/\sqrt{2n}}) \\ &= \exp\left[-\left(t\sqrt{\frac{2}{n}}\right)\left(\frac{n}{2}\right)\right]\left(1 - 2\frac{t}{\sqrt{2n}}\right)^{-n/2}, \qquad t < \frac{\sqrt{2n}}{2}. \end{aligned}$$

This may be written in the form

$$M(t; n) = \left(e^{t\sqrt{2/n}} - t\sqrt{\frac{2}{n}}\, e^{t\sqrt{2/n}}\right)^{-n/2}, \qquad t < \sqrt{\frac{n}{2}}.$$

In accordance with Taylor's formula, there exists a number $\xi(n)$, between 0 and $t\sqrt{2/n}$, such that

$$e^{t\sqrt{2/n}} = 1 + t\sqrt{\frac{2}{n}} + \frac{1}{2}\left(t\sqrt{\frac{2}{n}}\right)^2 + \frac{e^{\xi(n)}}{6}\left(t\sqrt{\frac{2}{n}}\right)^3.$$

If this sum is substituted for $e^{t\sqrt{2/n}}$ in the last expression for $M(t; n)$, it is seen that

$$M(t; n) = \left(1 - \frac{t^2}{n} + \frac{\psi(n)}{n}\right)^{-n/2},$$

where

$$\psi(n) = \frac{\sqrt{2}t^3 e^{\xi(n)}}{3\sqrt{n}} - \frac{\sqrt{2}t^3}{\sqrt{n}} - \frac{2t^4 e^{\xi(n)}}{3n}.$$

Since $\xi(n) \to 0$ as $n \to \infty$, then $\lim \psi(n) = 0$ for every fixed value of t. In accordance with the limit proposition cited earlier in this section, we have

$$\lim_{n \to \infty} M(t; n) = e^{t^2/2}$$

for all real values of t. That is, the random variable $Y_n = (Z_n - n)/\sqrt{2n}$ has a limiting normal distribution with mean zero and variance 1.

EXERCISES

5.11. Let X_n have a gamma distribution with parameter $\alpha = n$ and β, where β is not a function of n. Let $Y_n = X_n/n$. Find the limiting distribution of Y_n.

5.12. Let Z_n be $\chi^2(n)$ and let $W_n = Z_n/n^2$. Find the limiting distribution of W_n.

5.13. Let X be $\chi^2(50)$. Approximate Pr $(40 < X < 60)$.

Theorem 3. *Let $X_1, X_2, \ldots, X_n$ denote the items of a random sample from a distribution that has mean μ and positive variance σ^2. Then the random variable $Y_n = \left(\sum_1^n X_i - n\mu\right)/\sqrt{n}\sigma = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ has a limiting distribution that is normal with mean zero and variance 1.*

Proof. In the modification of the proof, we assume the existence of the moment-generating function $M(t) = E(e^{tX})$, $-h < t < h$, of the distribution. However, this proof is essentially the same one that would be given for this theorem in a more advanced course by replacing the moment-generating function by the characteristic function $\varphi(t) = E(e^{itX})$.

The function

$$m(t) = E[e^{t(X-\mu)}] = e^{-\mu t}M(t)$$

also exists for $-h < t < h$. Since $m(t)$ is the moment-generating function for $X - \mu$, it must follow that $m(0) = 1$, $m'(0) = E(X - \mu) = 0$, and $m''(0) = E[(X - \mu)^2] = \sigma^2$. By Taylor's formula there exists a number ξ between 0 and t such that

$$\begin{aligned} m(t) &= m(0) + m'(0)t + \frac{m''(\xi)t^2}{2} \\ &= 1 + \frac{m''(\xi)t^2}{2}. \end{aligned}$$

If $\sigma^2t^2/2$ is added and subtracted, then

$$m(t) = 1 + \frac{\sigma^2t^2}{2} + \frac{[m''(\xi) - \sigma^2]t^2}{2}.$$

Next consider $M(t; n)$, where

$$\begin{aligned} M(t; n) &= E\left[\exp\left(t\frac{\sum X_i - n\mu}{\sigma\sqrt{n}}\right)\right] \\ &= E\left[\exp\left(t\frac{X_1 - \mu}{\sigma\sqrt{n}}\right)\exp\left(t\frac{X_2 - \mu}{\sigma\sqrt{n}}\right)\cdots\exp\left(t\frac{X_n - \mu}{\sigma\sqrt{n}}\right)\right] \\ &= E\left[\exp\left(t\frac{X_1 - \mu}{\sigma\sqrt{n}}\right)\right]\cdots E\left[\exp\left(t\frac{X_n - \mu}{\sigma\sqrt{n}}\right)\right] \\ &= \left\{E\left[\exp\left(t\frac{X - \mu}{\sigma\sqrt{n}}\right)\right]\right\}^n \\ &= \left[m\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n, \qquad -h < \frac{t}{\sigma\sqrt{n}} < h. \end{aligned}$$

In $m(t)$, replace t by $t/\sigma\sqrt{n}$ to obtain

$$m\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \frac{t^2}{2n} + \frac{[m''(\xi) - \sigma^2]t^2}{2n\sigma^2},$$

where now ξ is between 0 and $t/\sigma\sqrt{n}$ with $-h\sigma\sqrt{n} < t < h\sigma\sqrt{n}$. Accordingly,

$$M(t; n) = \left\{1 + \frac{t^2}{2n} + \frac{[m''(\xi) - \sigma^2]t^2}{2n\sigma^2}\right\}^n.$$

Since $m''(t)$ is continuous at $t = 0$ and since $\xi \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} [m''(\xi) - \sigma^2] = 0.$$

The limit proposition cited in Section 5.3 shows that

$$\lim_{n\to\infty} M(t; n) = e^{t^2/2}$$

for all real values of t. This proves that the random variable $Y_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ has a limiting normal distribution with mean zero and variance 1.

Example 2. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a distribution that is $b(1, p)$. Here $\mu = p$, $\sigma^2 = p(1 - p)$, and $M(t)$ exists for all real values of t. If $Y_n = X_1 + \cdots + X_n$, it is known that Y_n is $b(n, p)$. Calculation of probabilities concerning Y_n, when we do not use the Poisson approximation, can be greatly simplified by making use of the fact that $(Y_n - np)/\sqrt{np(1 - p)} = \sqrt{n}(\bar{X}_n - p)/\sqrt{p(1 - p)} = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ has a limiting distribution that is normal with mean zero and variance 1. Let $n = 100$ and $p = \frac{1}{2}$, and suppose that we wish to compute Pr $(Y = 48, 49, 50, 51, 52)$. Since Y is a random variable of the discrete type, the events $Y = 48, 49, 50, 51, 52$ and $47.5 < Y < 52.5$ are equivalent. That is, Pr $(Y = 48, 49, 50, 51, 52) =$ Pr $(47.5 < Y < 52.5)$. Since $np = 50$ and $np(1 - p) = 25$, the latter probability may be written

$$\Pr(47.5 < Y < 52.5) = \Pr\left(\frac{47.5 - 50}{5} < \frac{Y - 50}{5} < \frac{52.5 - 50}{5}\right)$$

$$= \Pr\left(-0.5 < \frac{Y - 50}{5} < 0.5\right).$$

5.21. Let $\bar{X}$ denote the mean of a random sample of size 128 from a gamma distribution with $\alpha = 2$ and $\beta = 4$. Approximate $\Pr(7 < \bar{X} < 9)$.

5.22. Let Y be $b(72, \frac{1}{3})$. Approximate $\Pr(22 \le Y \le 28)$.

EXERCISES

5.20. Let $\bar{X}$ denote the mean of a random sample of size 100 from a distribution that is $\chi^2(50)$. Compute an approximate value of $\Pr(49 < \bar{X} < 51)$.

5.24. Let Y denote the sum of the items of a random sample of size 12 from a distribution having p.d.f. $f(x) = \frac{1}{6}$, $x = 1, 2, 3, 4, 5, 6$, zero elsewhere. Compute an approximate value of $\Pr(36 \le Y \le 48)$. *Hint.* Since the event of interest is $Y = 36, 37, \ldots, 48$, rewrite the probability as $\Pr(35.5 < Y < 48.5)$.

5.25. Let Y be $b(400, \frac{1}{5})$. Compute an approximate value of $\Pr(0.25 < Y/n)$.

Since $(Y - 50)/5$ has an approximate normal distribution with mean zero and variance 1, Table III shows this probability to be approximately 0.382.

The convention of selecting the event $47.5 < Y < 52.5$, instead of, say, $47.8 < Y < 52.3$, as the event equivalent to the event $Y = 48, 49, 50, 51, 52$ seems to have originated in the following manner: The probability, $\Pr(Y = 48, 49, 50, 51, 52)$, can be interpreted as the sum of five rectangular areas where the rectangles have bases 1 but the heights are, respectively, $\Pr(Y = 48), \ldots, \Pr(Y = 52)$. If these rectangles are so located that the midpoints of their bases are, respectively, at the points $48, 49, \ldots, 52$ on a horizontal axis, then in approximating the sum of these areas by an area bounded by the horizontal axis, the graph of a normal p.d.f., and two ordinates, it seems reasonable to take the two ordinates at the points 47.5 and 52.5.