Chapter two concept of limit point

$suppose\ f\ is\ defined\ at\ all\ points\ in\ some\ nbhd\ of\ a\ point\ z_0$ $, by\ the\ statment$

$that$

$\lim_{z \to z_0} f(z) = w_0 \quad \dots (*)$

$we\ mean\ that\ w = f(z)\ can\ be\ made\ arbitrary\ close\ to\ w_0\ provided\ z$

$is\ sufficiently\ close\ to\ z_0\ \ and\ distinct\ from\ it.$

$mathematically: -$

$this\ means\ for\ each\ \epsilon > 0\ arbitarly\ small\ there\ exist\ \delta > 0\ \ni$

$|f(z) - w_0| < \epsilon\ if\ 0 < |z - z_0| < \delta\ note\ that\ \delta\ depends\ on\ \epsilon\ and\ z_0\ both$

i.e. $\delta \equiv \delta(\epsilon, z_o)$.

$Geometrically: -$

$this\ means\ for\ each\ \epsilon - nbhd\ |w - w_0| < \epsilon\ of\ w_0\ there\ exists\ adeleted$

$\delta - nbhd\ of\ z_0\ such\ that\ each\ point\ of\ this\ a\ deleted\ \delta$
$- nbhd\ of\ z_0\ such\ that\ each\ point\ of\ this\ nbhd\ has\ an\ \ image\ lying\ in\ the\ \epsilon$
$- nbhd\ of\ w_0\ since\ (*) holds\ for\ each\ point\ of\ the\ nbhd\ we\ have$

1. $z\ goes\ to\ z_0\ in\ any\ path\ .$
2. $f\ should\ be\ define\ at\ all\ point$
3. $there\ must\ be\ the\ connectedness$

$notice\ that\ nbhd\ of\ z_0\ comes\ into\ enistance\ if\ z_0\ is\ an\ interior\ point\ .$

$(in\ particular\ in\ the\ case\ of\ an\ open\ set)$

$by\ the\ uniform\ limit\ we\ mean\ \delta\ depends\ only\ on\ \epsilon. (i.e\ independent\ of\ z_0)$

$uniqueness\ of\ the\ limit\ : -$

$suppose\ \lim_{z \to z_0} f(z) = w_0\ ,\ \lim_{z \to z_0} f(z) = w_1\ ,$

$\Rightarrow w_0 = w_1$

$example \;\; : -if\ f(z) = \frac{z}{\bar{z}}\ \ then$

$\lim_{z\to 0} f(z) \ldots\ldots (*)\quad does\ not\ exist\ .$

$for\ , if\ it\ did\ exist\ , it\ could\ be\ found\ by\ letting\ the\ point\ z =$
$(x,y) approach\ the\ origin\ in\ any\ manner\ . but\ when\ z = (x,0) is\ a\ non -$
$zero\ point\ on\ the\ real\ axis\ .$

$$f(z) = \frac{x + i0}{x - i0} = 1$$

$and\ when\ z = (0,y) is\ a\ non - zero\ point\ on\ the\ imaginary\ axis\ ,$

$$f(z) = \frac{0 + iy}{0 - iy} = -1$$

$thus\ , by\ letting\ z$ approach the origin a long $the\ real\ axis\ we\ would\ find\ that\ the$

$desired\ limit\ is$ 1.

$an\ approach\ a\ long\ the\ imaginary\ axis\ would\ , on\ the\ other\ hand\ , yield\ the\ limit$
$-1\ , since\ a\ limit\ is\ unique\ , we\ must\ conclude\ that\ limit\ (*) does\ not\ exist.$

$\boldsymbol{theorem}: -$

$let\ f(z) = u + iv\ , z_0 = x_0 + iy_0\ , w_0 = u_0 + iv_0$

$then$

$$\lim_{z\to z_0} f(z) = w_0\ \ if\ and\ only\ if\ \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$$

and

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

$remarks\ : -$

1. $\lim_{z\to z_0} f(z) = w_0 \Leftrightarrow \lim_{z\to z_0} \overline{f(z)} = \overline{w_0}$

$proof:-$

$$\lim_{z\to z_0} f(z) = w_0 \Leftrightarrow for\ each\ \in > 0, \exists\ \delta > 0\ \ni 0 < |z - z_0| < \delta$$

$$\Leftrightarrow |f(z) - w_0| < \epsilon$$

$$\Leftrightarrow |\overline{f(z) - w_0}| < \epsilon$$

$$\Leftrightarrow \left|\overline{f(z)} - \overline{w_0}\right| < \epsilon$$

$$\therefore \lim_{z\to z_0} \overline{f(z)} = \overline{w_0}$$

2. $let\ c\ be\ any\ constant\ complex\ number\ then\ \lim_{z\to z_0} c = c$
3. $\lim_{z\to z_0} z^n = {z_0}^n$
4. $\lim_{z\to z_0} \bar{z}^n = \bar{z_0}^n\ (or\ \lim_{z\to z_0} \overline{z^n} = \overline{{z_0}^n})$

$theorem:-$

$suppose\ that$

$$\lim_{z\to z_0} f(z) = L\quad, \lim_{z\to z_0} g(z) = m\ \ ,then$$

1. $\lim_{z\to z_0}[f(z) \pm g(z)] = L \pm m$
2. $\lim_{z\to z_0}[f(z).g(z)] = L.m$
3. $\lim_{z\to z_0}\left[\frac{f(z)}{g(z)}\right] = \frac{L}{m}\ \ if\ g(z) \neq 0$