ANGLE MODULATION

1. PHASE AND FREQUENCY MODULATION

For angle modulation, the modulated carrier is represented by

$$x_c(t) = A\, cos[\omega_c t + \phi(t)] \tag{1.1}$$

Where A and ω_c are constants and the phase angle $\phi(t)$ is a function of the message signal $m(t)$. Equation (1.1) can be written as

$$x_c(t) = A\, cos\, \theta(t)$$

where $$\theta(t) = \omega_c t + \phi(t)$$

The instantaneous radian frequency of $x_c(t)$, denoted by ω_i is

$$\omega_i = \frac{d\,\theta(t)}{dt} = \omega_c + \frac{d\,\phi(t)}{dt} \tag{1.2}$$

$\phi(t)$ = instantaneous phase deviation.

$\frac{d\,\phi(t)}{dt} =$ instantaneous frequancy deviation.

The *maximum* (or *peak) radian frequency deviation* of the angle-modulated signal ($\Delta\omega$) is given by

$$\Delta\omega = |\omega_i - \omega_c|_{max} \tag{1.3}$$

In *phase modulation* (*PM*) the instantaneous phase deviation of the carrier is proportional to the message signal; that is,

$$\phi(t) = k_p m(t) \tag{1.4}$$

where k_p is the *phase deviation constant*, expressed in radians per unit of $m(t)$.
In *frequency modulation* (*FM*), the instantaneous frequency deviation of the carrier is proportional to the message signal; that is,

$$\frac{d\,\phi(t)}{dt} = k_f m(t) \tag{1.5a}$$

or $$\phi(t) = k_f \int_{-\infty}^{t} m(\lambda)\, d\lambda \tag{1.5b}$$

thus, we can express the angle-modulated signal as

$$x_{PM}(t) = A\, cos\, [\omega_c t + k_P m(t)] \tag{1.6}$$

$$x_{FM}(t) = A\, cos\left[\omega_c t + k_f \int_{-\infty}^{t} m(\lambda)\, d\lambda\right] \tag{1.7}$$

From Eq.(1.2),we have

$$\omega_i = \omega_c + k_p \frac{d\, m(t)}{dt} \qquad for\ PM \tag{1.8}$$

$$\omega_i = \omega_c + k_f m(t) \qquad for\ FM \tag{1.9}$$

Thus, in *PM*, the instantaneous frequency ω_i varies linearly with the derivative of the modulating signal, and in *FM*, ω_i varies linearly with the modulating signal. Figure (1.1) illustrates *AM*, *FM*, and *PM* waveforms produced by a sinusoidal message waveform.

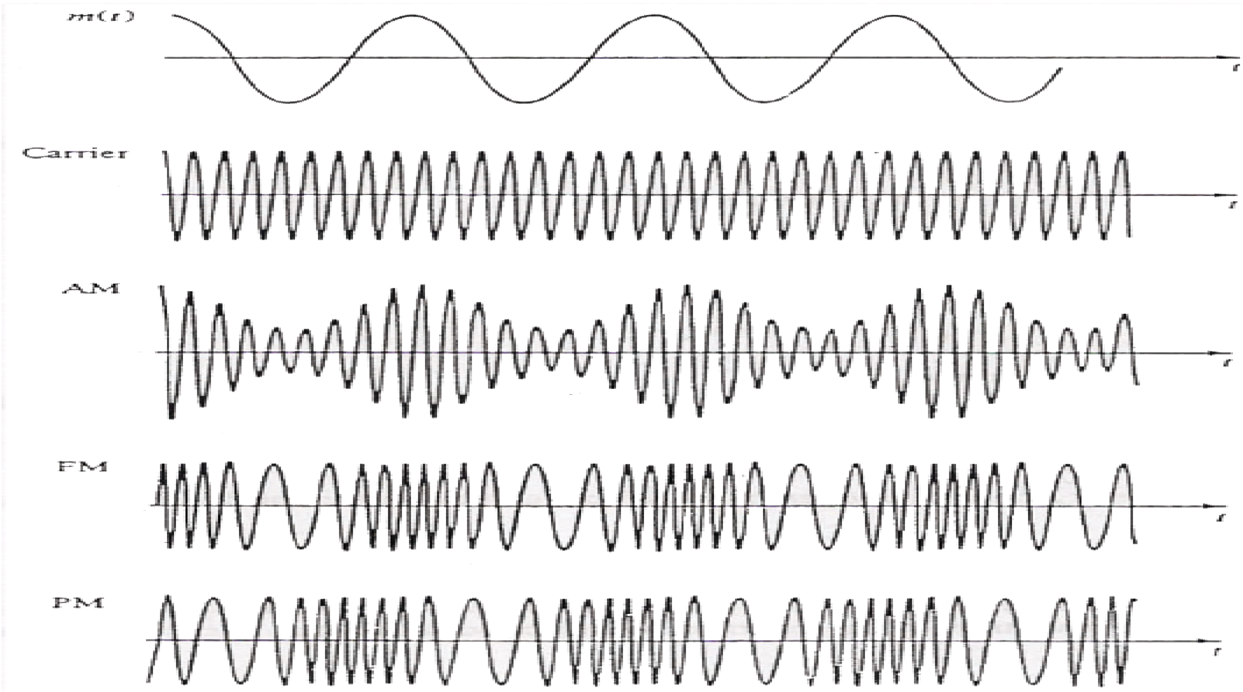


Fig.(1.1)

2. FOURIER SPECTRA OF ANGLE-MODULATED SIGNALS

An angle-modulated carrier can be represented in exponential form by writing Eq.(1.1) as

$$x_c(t) = Re\left(A\, e^{j(\omega_c t + \phi(t))}\right) = Re\left(A\, e^{j\omega_c t} e^{j\phi(t)}\right) \tag{2.1}$$

Expanding $e^{j\phi(t)}$in a power series yields

$$x_c(t) = Re\left\{Ae^{j\omega_c t}\left[1 + j\phi(t) - \frac{\phi^2(t)}{2!} - \cdots + j^n \frac{\phi^n(t)}{n!} + \cdots\right]\right\}$$

$$= A\left[cos\ \omega_c t - \phi(t) sin\omega_c t - \frac{\phi^2(t)}{2!} cos\omega_c t + \frac{\phi^3(t)}{3!} sin\omega_c t + \cdots\right] \tag{2.2}$$

Thus the angle-modulated signal consists of an unmodulated carrier plus various amplitude-modulated terms, such as $\phi(t) sin\ \omega_c t$, $\phi^2(t) cos\ \omega_c t$, $\phi^3(t) sin\ \omega_c t$ ... , etc. Hence its Fourier spectrum consists of an unmodulated carrier plus spectra of $\phi(t)$, $\phi^2(t)$, $\phi^3(t)$,... , etc., centered at ω_c.

It is clear that the Fourier spectrum of an angle-modulated signal is not related to the message signal spectrum in any simple way, as was the case in *AM*.

3. NARROWBAND ANGLE MODULATION

If $|\phi(t)|_{max} \ll 1$, then Eq. (2.2) can be approximated by [neglecting all higher-power terms of $\phi(t)$]

$$x_c(t) \approx Acos\omega_c t - A\phi(t)sin\omega_c t \qquad (3.1)$$

$x_c(t)$ in Eq.(3.1) is called the *narrowband* (*NB*) angle-modulated signal. Thus,

$$x_{NBPM}(t) \approx Acos\omega_c t - Ak_p\, m(t)sin\omega_c t \qquad (3.2)$$

$$x_{NBFM}(t) \approx Acos\omega_c t - A\left[k_f \int_{-\infty}^{t} m(\lambda)\, d\lambda\right] sin\omega_c t \qquad (3.3)$$

Equation (3.1) indicates that a narrowband angle-modulated signal contains an unmodulated carrier plus a term in which $\phi(t)$ [a function of $m(t)$] multiplies a $\pi/2$ (*rad*) phase-shifted carrier. This multiplication generates a pair of sidebands, and if $\phi(t)$ has a-bandwidth W_B, the bandwidth of an *NB* angle-modulated signal is $2W_B$.

4. SINUSOIDAL (OR TONE), MODULATION

If the message signal $m(t)$ is a pure sinusoid, that is,

$$m(t) = \begin{cases} a_m sin\omega_m t & for\ PM \\ a_m cos\omega_m t & for\ FM \end{cases}$$

then Eqs. (1.4) and (1.5b) both give

$$\phi(t) = \beta\ sin\omega_m t \qquad (4.1)$$

Where

$$\beta = \begin{cases} k_p a_m & for\ PM \\ \dfrac{k_f a_m}{\omega_m} & for\ FM \end{cases} \qquad (4.2)$$

The parameter β is known as the <u>modulation index</u> for angle modulation and is the maximum value of phase deviation for both *PM* and *FM*. Note that β is defined only for sinusoidal modulation and it can be expressed as

$$\beta = \frac{\Delta\omega}{\omega_m} \qquad (4.3)$$

Substituting Eq. (4.1) into Eq. (1.1), we obtain

$$x_c(t) = A\ cos[\omega_c t + \beta\ sin\omega_m t] \qquad (4.4)$$

which can be expressed as

$$x_c(t) = A\ Re(e^{j\,\omega_c t} e^{j\beta sin\ \omega_m t}) \qquad (4.5)$$

The function $e^{j\beta sin\ \omega_m t}$ is clearly a periodic function with period $T_m = 2\pi/\omega_m$. It therefore has a Fourier series representation

$$e^{j\beta sin\ \omega_m t} = \sum_{n=-\infty}^{\infty} c_n e^{jn\,\omega_m t}$$

The Fourier coefficients c_n can be found to be

$$c_n = \frac{\omega_m}{2\pi} \int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j\beta sin\ \omega_m t}\ e^{-jn\,\omega_m t} dt$$

Setting $\omega_m t = x$, we have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta sinx - nx)}\, dx = J_n(\beta)$$

Where $J_n(\beta)$is the Bessel function of the first kind of order *n* and argument *β*. These functions are plotted in Fig.(4.1) as a function of *n* for various values of *β*.

Note that :

1. $J_{-n}(\beta) = (-1)^n J_n(\beta)$
2. $J_{n-1}(\beta) + J_{n+1}(\beta) = \frac{2n}{\beta} J_n(\beta)$
3. $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$

Thus

$$e^{j\beta sin\,\omega_m t} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\,\omega_m t} \qquad (4.6)$$

Substituting Eq.(4.6) into Eq.(4.5), we obtain

$$x_c(t) = A\,Re\left[e^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\,\omega_m t}\right]$$
$$= A\,Re\left[\sum_{n=-\infty}^{\infty} J_n(\beta) e^{j(\omega_c + n\omega_m)t}\right]$$

Taking the real part yield

$$x_c(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) cos(\omega_c + n\omega_m)t \qquad (4.7)$$

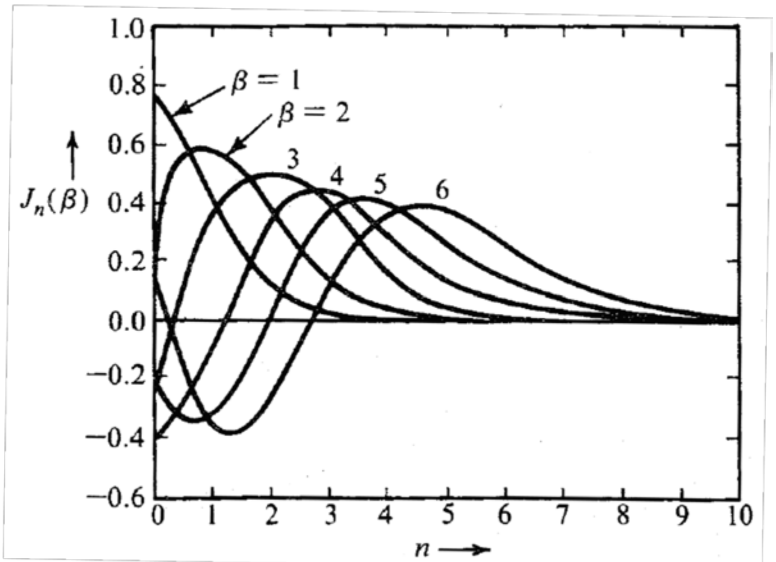


Fig.(4.1)

We observe that;

1. The spectrum consists of a carrier-frequency component plus an infinite number of sideband components at frequencies $\omega_c \pm n\omega_m$ (n = 1, 2, 3, ...).
2. The relative amplitudes of the spectral lines depend on the value of $J_n(\beta)$, and the value of $J_n(\beta)$ becomes very small for large values of n.
3. The number of significant spectral lines (that is, having appreciable relative amplitude) is a function of the modulation index β. With $\beta \ll 1$, only J_0 and J_1 are significant, so the spectrum will consist of carrier and two sideband lines. But if $\beta \gg 1$, there will be many sideband lines. Figure (4.2) shows the amplitude spectra of angle-modulated signals for several values of β.

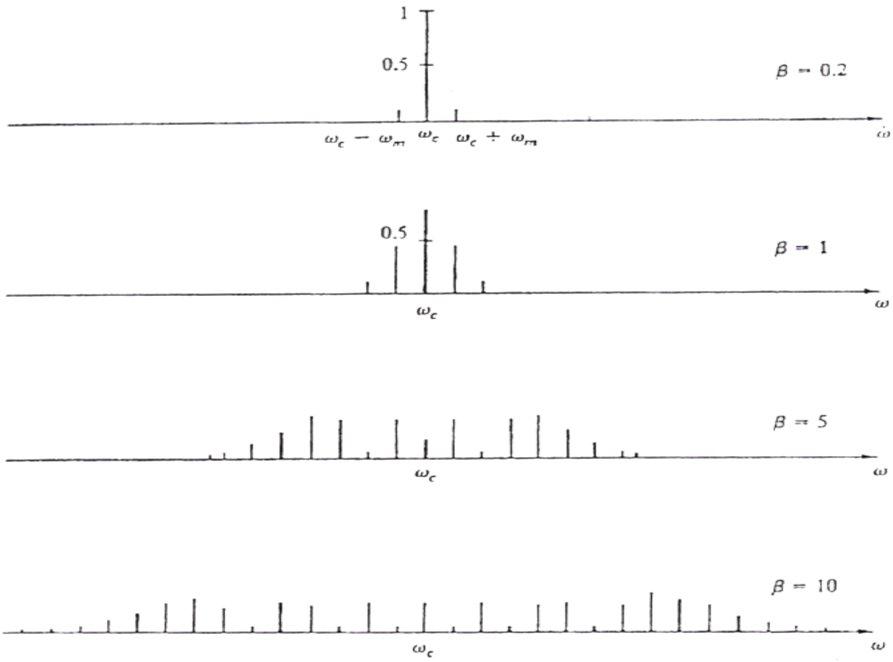


Fig.(4.2)

<u>Example 4.1:</u> For the angle-modulated signal,

$$x_c(t) = 10\, cos\, [2\pi(10^6)t + 10\, sin\, 2\pi(10^3)t]$$

Find $m(t)$ if;

(a) $x_c(t)$ is a *PM* signal with $k_p = 10$.

(b) $x_c(t)$ is a *FM* signal with $k_f = 10\pi$.

<u>Sol.</u>

(a) $k_p m(t) = 10\, sin\, 2\pi(10^3)t$

$\therefore\ m(t) = sin\, 2\pi(10^3)t$

(b) $k_f \int_{-\infty}^{t} m(\lambda)\, d\lambda = 10\, sin\, 2\pi(10^3)t$

$$\int_{-\infty}^{t} m(\lambda)\, d\lambda = \frac{10}{10\pi}\, sin\, 2\pi(10^3)t$$

$$\therefore \quad m(t) = \frac{d}{dt}\left(\frac{1}{\pi} \sin 2\pi(10^3)\, t\right) = 2(10^3) cos\, 2\pi(10^3)t$$

Note that $\theta(t) = \omega_c t + \phi(t) = 2\pi(10^6)t + 10\, sin\, 2\pi(10^3)t$

and $\phi(t) = 10\, sin\, 2\pi(10^3)t$
now $\acute{\phi}(t) = 20\,(\pi 10^3)\, cos\, 2\pi(10^3)t$
thus, the maximum phase deviation is
$|\phi(t)|_{max} = 10\, rad$
And the maximum frequency deviation is
$\Delta\omega = \left|\acute{\phi}(t)\right|_{max} = 20\,(\pi 10^3)\;\; rad/s = 10\, KHz$

5. BANDWIDTH OF ANGLE MODULATED SIGNALS

5.1 Sinusoidal Modulation

The bandwidth of angle-modulated signal with sinusoidal modulation depends on β and ω_m. From mathematic relations;

$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = J_0^2(\beta) + 2(J_1^2(\beta) + J_2^2(\beta) + \cdots) = 1$$

This property will help us to find the power of angle modulated signal as

$$P = \left(A J_0(\beta)/\sqrt{2}\right)^2 + 2\left[\left(A J_1(\beta)/\sqrt{2}\right)^2 + \left(A J_2(\beta)/\sqrt{2}\right)^2 + \cdots\right]$$
$$= \frac{A^2}{2}\,[J_0^2(\beta) + 2(J_1^2(\beta) + J_2^2(\beta) + \cdots)] = \frac{A^2}{2} \times 1 = \frac{A^2}{2}$$

Now we can define the bandwidth of angle modulated signal as the band of frequencies or harmonics which consists about of 98% of the normalized total signal power and is given by;

$$W_B \approx 2(\beta + 1)\omega_m \tag{5.1}$$

When $\beta \ll 1$, the signal is an *NB* angle-modulated signal and its bandwidth is approximately equal $2\omega_m$. Usually a value of $\beta < 0.2$ is taken to be sufficient to satisfy this condition.

Let us consider β =1, then $J_0(1) = 0.7652$, $J_1(1) = 0.44$, $J_2(1) = 0.1149$, $J_3(1) = 0.002477$, then the power considered in the terms of n = 0,1, and 2

$$P = \frac{1}{2}\left[{J_0}^2(1) + 2\left({J_1}^2(1) + {J_2}^2(1)\right)\right]$$
$$= \frac{1}{2}[0.7652^2 + 2(0.44^2 + 0.1149^2)] = 0.495$$

The sum of P for n=2 is 99% of the total power, which is 0.5. If the amplitude $J_n(\beta) < 0.1$, it can be neglected.

5.2 Arbitrary Modulation:

For an angle-modulated signal with an arbitrary modulating signal $m(t)$ band-limited to ω_M rad/s, we define the *deviation ratio D* as

$$D = \frac{maximum\ frequency\ deviation}{bandwidth\ of\ m(t)} = \frac{\Delta\omega}{\omega_M} \tag{5.2}$$

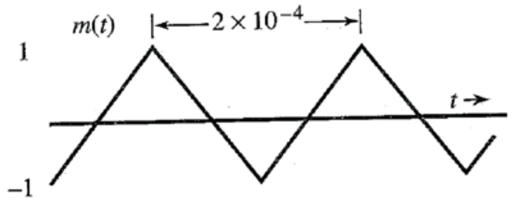
The deviation ratio D plays the same role for arbitrary modulation as the modulation index β plays for sinusoidal modulation. Replacing β by D and ω_m by ω_M in Eq. (5.1), we have

$$W_B \approx 2(D + 1)\omega_M \approx 2(\Delta\omega + \omega_M) \tag{5.3}$$

This expression for bandwidth is generally referred to as *Carson's rule*. If $D \ll 1$, the bandwidth is approximately $2\omega_M$,and the signal is known as a *narrowband* (*NB*) angle-modulated signal. If $D \gg 1$, the bandwidth is approximately $2D\omega_M = 2\Delta\omega$, which is twice the peak frequency deviation. Such a signal is called a *wideband* (*WB*) angle-modulated signal.

Example 5.1:

(a) Estimate B_{FM} and B_{PM} for the modulating signal $m(t)$ for $k_f = 2\pi\times10^5$ and $k_p = 5\pi$.
(b) Repeat the problem if the amplitude of $m(t)$ is doubled.



Sol.

(a) The peak amplitude of $m(t)$ is unity. Hence, $m(t)|_{max} = 1$. We now determine the essential bandwidth B of $m(t)$. The Fourier series for this periodic signal is given by

$$m(t) = \sum_n a_n \cos n\omega_o t \qquad \omega_o = \frac{2\pi}{2\times10^{-4}} = 10^4\pi$$

Where

$$a_n = \begin{cases} \dfrac{-8}{\pi^2 n^2} & n \ odd \\ 0 & n\ even \end{cases}$$

It can be seen that the harmonic amplitudes decrease rapidly with n. The third and fifth harmonic powers are 1.21 and 0.16%, respectively, of the fundamental component power. Hence, we are justified in assuming the essential bandwidth of $m(t)$ as the frequency of the third harmonic, that is, $3(10^4/2)$ Hz. Thus,

$$f_M = 15 \text{ kHz}$$

For *FM*:

$$\Delta f = \frac{\Delta\omega}{2\pi} = \frac{1}{2\pi}k_f m(t)|_{max} = \frac{1}{2\pi}(2\pi\times10^5)\times 1 = 100\ kHz$$

and

$$B_{FM} = 2(\Delta f + f_M) = 230\ kHz$$

The deviation ratio D is given by

$$D = \frac{\Delta f}{f_M} = \frac{100}{15}$$

For *PM*: the peak amplitude of $\acute{m}(t)$ is 2×10^4 and

$$\Delta f = \frac{1}{2\pi}k_p \acute{m}(t)|_{max} = 50\ kHz$$

Hence,

$$B_{PM} = 2(\Delta f + f_M) = 130\ kHz$$

and

$$D = \frac{\Delta f}{f_M} = \frac{50}{15}$$

(b) Doubling *m*(*t*) doubles its peak value. Hence, $m(t)|_{max}$ =2. But its bandwidth is unchanged ($f_M = 15\ kHz$).

For *FM*:

$$\Delta f = \frac{1}{2\pi} k_f m(t)|_{max} = \frac{1}{2\pi}(2\pi \times 10^5) \times 2 = 200\ kHz$$

and

$$B_{FM} = 2(\Delta f + f_M) = 430\ kHz$$

The deviation ratio *D* is given by

$$D = \frac{\Delta f}{f_M} = \frac{200}{15}$$

For *PM*: Doubling *m*(*t*) doubles its derivative so that now $\acute{m}(t)|_{max} = 4\times10^4$, and

$$\Delta f = \frac{1}{2\pi} k_p \acute{m}(t)|_{max} = 100kHz$$

Hence,

$$B_{PM} = 2(\Delta f + f_M) = 230\ kHz$$

and

$$D = \frac{\Delta f}{f_M} = \frac{100}{15}$$

Observe that doubling the signal amplitude roughly doubles the bandwidth of both *FM* and *PM* waveforms.

If *m*(*t*) is time-expanded by a factor of 2; that is, if the period of *m*(*t*) is 4×10^{-4} ,then the signal spectral width (bandwidth) reduces by a factor of 2. We can verify this by observing that the fundamental frequency is now 2.5 *kHz*, and its third harmonic is 7.5 *kHz*. Hence, f_M = 7.5 *kHz*, which is half the previous bandwidth. Moreover, time expansion does not affect the peak amplitude so that $m(t)|_{max} = 1$. However, $\acute{m}(t)|_{max}$ is halved, that is, $\acute{m}(t)|_{max}$ = 10, 000.

For *FM*:

$$\Delta f = \frac{1}{2\pi} k_f m(t)|_{max} = 100\ kHz$$

and

$$B_{FM} = 2(\Delta f + f_M) = 2(100 + 7.5) = 215\ kHz$$

For *PM*:

$$\Delta f = \frac{1}{2\pi} k_p \acute{m}(t)|_{max} = 25\ kHz$$

Hence,

$$B_{PM} = 2(\Delta f + f_M) = 65\ kHz$$

Note that time expansion of $\boldsymbol{m(t)}$ has very little effect on the FM band width, but it halves the PM bandwidth. This verifies that the PM spectrum is strongly dependent on the spectrum of $\boldsymbol{m(t)}$.

6. GENERATION OF ANGLE-MODULUED SIGNALS

6.1 Narrowband Angle-Modulated Signals:

The generation of narrowband angle-modulated signals is easily accomplished in view of Eq.(3.2) and (3.3). This is illustrated in Fig.(6.1)

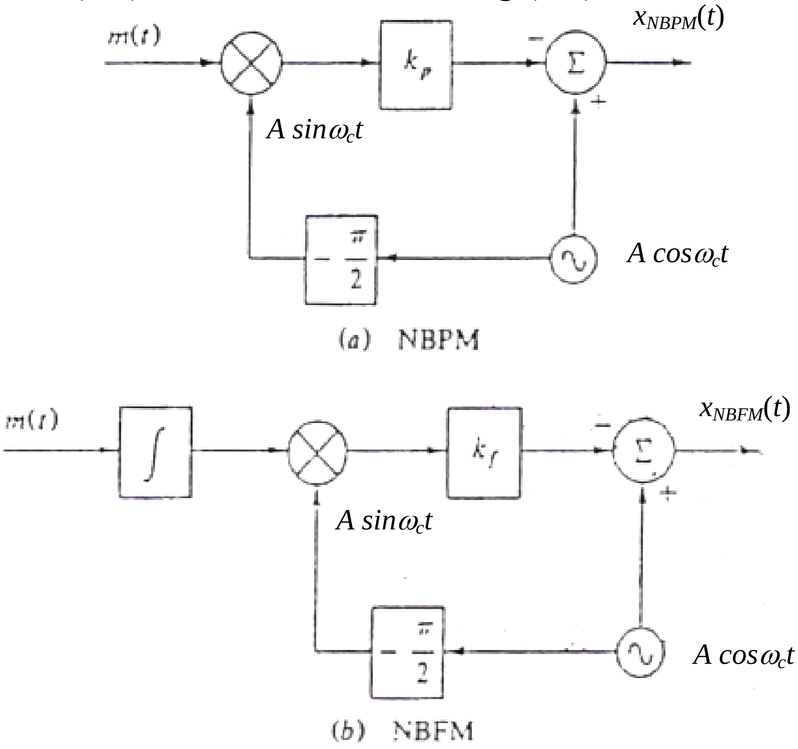


Fig.(6.1)

6.2 Wideband Angle-Modulated Signals:

There are two methods of generating wideband (*WB*) angle-modulated signals; the indirect method and the direct method.

6.2.1 Indirect Method of Armstrong

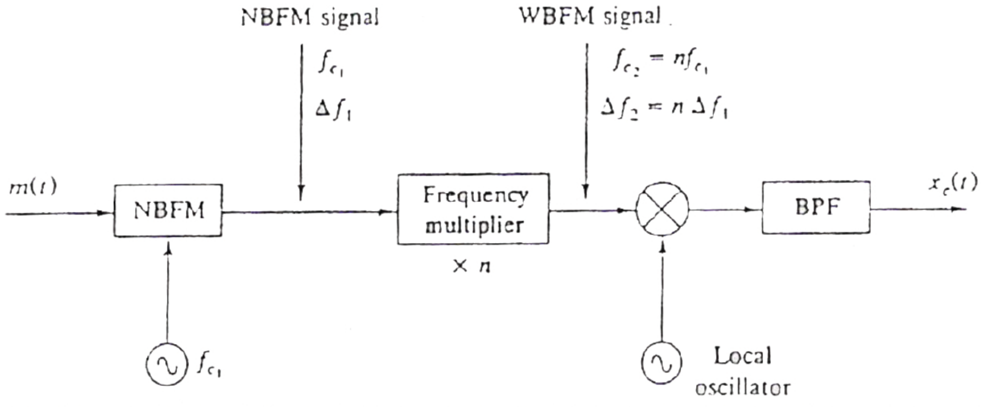
In this method, an *NB* angle-modulated signal is produced first and then converted to a *WB* angle-modulated signal by using frequency multipliers. The frequency multiplier multiplies the argument of the input sinusoid by *n*. Thus, if the input of a frequency multiplier is

$$x(t) = A\,cos[\omega_c t + \phi(t)]$$

Then the output of the frequency multiplier is

$$y(t) = A\,cos[n\omega_c t + n\phi(t)]$$

Use of frequency multiplication normally increases the carrier frequency to an impractically high value. To avoid this, a frequency conversion (using a mixer or *DSB* modulator) is necessary to shift the spectrum.



6.2.2 Direct Generation

In a voltage-controlled oscillator (*VCO*), the frequency is controlled by an external voltage. The oscillation frequency varies linearly with the control voltage. We can generate an *FM* wave by using the modulating signal $m(t)$ as a control signal.

$$\omega_i(t) = \omega_c + k_f\, m(t)$$

One can construct a *VCO* using variable reactive element (*C* or L) in resonant circuit of an oscillator. In *Hartley* or *Colpitt* oscillators, the frequency of oscillation is given by

$$\omega_o = \frac{1}{\sqrt{LC}}$$

If the capacitance C is varied by the modulating signal $m(t)$

$$C = C_o - km(t)$$

$$\omega_o = \frac{1}{\sqrt{LC_o\left[1 - \frac{km(t)}{C_o}\right]}} = \frac{1}{\sqrt{LC_o}\left[1 - \frac{km(t)}{C_o}\right]^{1/2}}$$

$$\approx \frac{1}{\sqrt{LC_o}}\left[1 + \frac{km(t)}{2C_o}\right] \qquad \frac{km(t)}{C_o} \ll 1$$

Where $(1+x)^n \approx 1 + nx \; for |x| \ll 1.$ Thus,

$$\omega_o = \omega_c\left[1 + \frac{km(t)}{2C_o}\right] \qquad \omega_c = \frac{1}{\sqrt{LC_o}}$$

$$\omega_o = \omega_c + k_f m(t) \qquad k_f = \frac{k\omega_c}{2C_o}$$

The main advantage of direct *FM* is that large frequency deviations are possible and thus less frequency multiplication is required. The major disadvantage is that the carrier frequency tends to drift and so additional circuitry is required for frequency stabilization.

7. DEMODULATION OF ANGLE-MODULATED SIGNALS

Demodulation of an *FM* signal requires a system that produces an output proportional to the instantaneous frequency deviation of the input signal. Such a system is called a *frequency discriminator*. If the input to an ideal discriminator is an angle-modulated signal

$$x_c(t) = A\,cos[\omega_c t + \phi(t)]$$

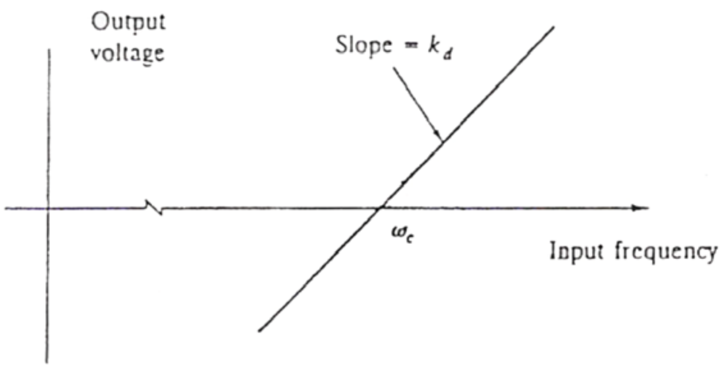
then the output of the discriminator is

$$y_d(t) = k_d \frac{d\phi(t)}{dt}$$

where k_d is the discriminator sensitivity. For *FM*

$$y_d(t) = k_d k_f m(t)$$

The characteristics of an ideal frequency discriminator are shown below



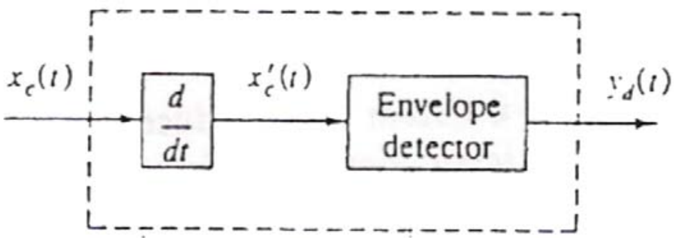
The frequency discriminator also can be used to demodulate *PM* signals. For *PM*, $\phi(t)$ is given by

$$\phi(t) = k_p m(t)$$

Then $y_d(t)$

$$y_d(t) = k_d k_p \frac{d\,m(t)}{dt}$$

Integration of the discriminator output yields a signal which is proportional to *m*(*t*). A demodulator for *PM* can therefore be implemented as an *FM* demodulator followed by an integrator. A simple approximation to the ideal discriminator is an ideal-differentiator followed by an envelope detector.



the output of the differentiator is

$$\acute{x}_c(t) = -A\left[\omega_c + \frac{d\phi(t)}{dt}\right] sin[\omega_c + \phi(t)]$$

The signal $\acute{x}_c(t)$ is both amplitude- and angle-modulated. The envelope of $\acute{x}_c(t)$is

$$A\left[\omega_c + \frac{d\phi(t)}{dt}\right]$$

The output of the envelop detector is

$$y_d(t) = \omega_i$$

Which is the instantaneous frequency of the $x_c(t)$.

8. NOISE IN ANGLE MODULATION SYSTEMS

The transmitted signal $X_c(t)$ has the form

$$X_c(t) = A_c\, cos[\omega_c t + \phi(t)] \qquad (8.1)$$

Where

$$\phi(t) = \begin{cases} k_p X(t) & for\ PM \\ k_f \int_{-\infty}^{t} X(\lambda)\, d\lambda & for\ FM \end{cases} \qquad (8.2)$$

Figure (8.1) shows a model for the angle demodulation system. The predetection filter bandwidth $B_T = 2(D+1)B$. The detector input is

$$Y_i(t) = X_c(t) + n_i(t)$$

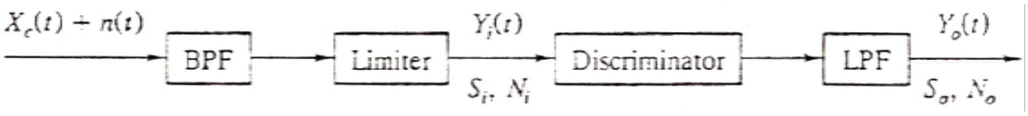


Fig.(8.1)

Where

$$n_i(t) = n_c(t)\, cos\omega_c t - n_s(t)\, sin\omega_c t$$

$$= v_n(t)\, cos[\omega_c t + \phi_n(t)]$$

The carrier amplitude remains constant, therefore

$$S_i = E[X_c^2(t)] = \frac{1}{2} A_c^2$$

and

$$N_i = \eta B_T$$

Hence,

$$\left(\frac{S}{N}\right)_i = \frac{A_c^2}{2\eta B_T} \qquad (8.3)$$

Which is independent of $X(t)$. $\left(\frac{S}{N}\right)_i$ is often called the *carrier-to-noise ratio* (*CNR*).

$$Y_i(t) = V(t)\, cos\,[\omega_c t + \theta(t)] \qquad (8.4)$$

Where

$$V(t) = \{[A_c cos\phi + v_n(t) cos\phi_n(t)]^2 + [A_c sin\phi + v_n(t) sin\phi_n(t)]^2\}^{1/2} \qquad (8.5)$$

and

$$\theta(t) = tan^{-1} \frac{A_c sin\phi(t) + v_n(t) sin\phi_n(t)}{A_c cos\phi(t) + v_n(t) cos\phi_n(t)} \qquad (8.6)$$

The limiter suppresses any amplitude variation $V(t)$. Hence, in angle modulation, *SNRs* are derived from consideration of $\theta(t)$ only. The detector is assumed to be ideal. The output of the detector is

$$Y_o(t) = \begin{cases} \theta(t) & for\ PM \\ \frac{d\,\theta(t)}{dt} & for\ FM \end{cases} \qquad (8.7)$$

Let

$$Y_i(t) = Re\left[Y(t)e^{j\omega_c t}\right] \tag{8.8}$$

where

$$Y(t) = A_c e^{j\phi(t)} + v_n e^{j\phi_n(t)} \tag{8.9}$$

For signal dominance case, $v_n \ll A_c$ for almost all t. from Fig.(8.2) the length L of arc AB is

$$L = Y(t)[\theta(t) - \phi(t)] \tag{8.10}$$

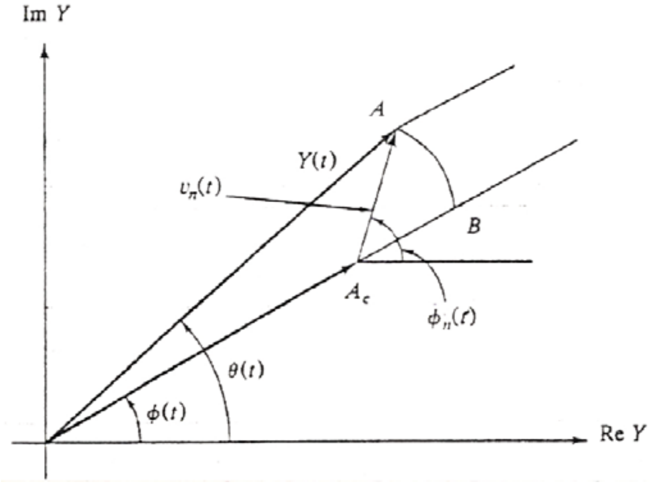


Fig.(8.2)

and

$$Y(t) = A_c + v_n(t)\, cos[\phi_n(t) - \phi(t)] \approx A_c \tag{8.11}$$

$$L \approx v_n(t)\, sin[\phi_n(t) - \phi(t)] \tag{8.12}$$

Hence, from Eq.(8.10), we obtain

$$\theta(t) \approx \phi(t) + \frac{v_n(t)}{A_c}\, sin[\phi_n(t) - \phi(t)] \tag{8.13}$$

Replacing $\phi_n(t) - \phi(t)$ with $\phi_n(t)$ will not affect the result. Thus

$$\theta(t) \approx \phi(t) + \frac{v_n(t)}{A_c}\, sin[\phi_n(t)]$$

$$\theta(t) \approx \phi(t) + \frac{n_s(t)}{A_c} \tag{8.14}$$

From Eqs.(8.7) and (8.2) the detector output is

$$Y_o(t) = \theta(t) = k_p X(t) + \frac{n_s(t)}{A_c} \qquad for\ PM \tag{8.15}$$

$$Y_o(t) = \frac{d\,\theta(t)}{dt} = k_f X(t) + \frac{\acute{n}_s(t)}{A_c} \qquad for\ FM \tag{8.16}$$

8.1 $(S/N)_o$ in *PM* systems

From Eq.(8.15)

$$S_o = E\left[k_p^2 X^2(t)\right] = k_p^2 E[X^2(t)] = k_p^2 S_X \tag{8.17}$$

$$N_o = E\left[\frac{1}{A_c^2} n_s^2(t)\right] = \frac{1}{A_c^2} E[n_s^2(t)] = \frac{1}{A_c^2}(2\eta B) \tag{8.18}$$

Hence,

$$\left(\frac{S}{N}\right)_o = \frac{k_p^2 A_c^2 S_X}{2\eta B} \tag{8.19}$$

Since,

$$\gamma = \frac{S_i}{\eta B} = \frac{A_c^2}{2\eta B} \tag{8.20}$$

Eq.(8.19) can be expressed as

$$\left(\frac{S}{N}\right)_o = k_p^2 S_X \gamma \tag{8.21}$$

8.2 (*S/N*)$_o$ in *FM* systems

From Eq.(8.16)

$$S_o = E\left[k_f^2 X^2(t)\right] = k_f^2 E[X^2(t)] = k_f^2 S_X \tag{8.22}$$

$$N_o = E\left[\frac{1}{A_c^2}[\acute{n}_s(t)]^2\right] = \frac{1}{A_c^2} E[[\acute{n}_s(t)]^2] \tag{8.23}$$

The *PSD* of $\acute{n}_s(t)$ is given by

$$S_{\acute{n}_s \acute{n}_s}(\omega) = \omega^2 S_{n_s n_s}(\omega) = \begin{cases} \omega^2 \eta & for\ |\omega| < W\ (= 2\pi B) \\ 0 & otherwise \end{cases} \tag{8.24}$$

Then

$$N_o = \frac{1}{A_c^2}\frac{1}{2\pi}\int_{-W}^{W} \omega^2 \eta \ d\omega = \frac{2}{3}\frac{\eta}{A_c^2}\frac{W^3}{2\pi} \tag{8.25}$$

Hence,

$$\left(\frac{S}{N}\right)_o = \frac{3A_c^2(2\pi)k_f^2 S_X}{2\eta W^3} \tag{8.26}$$

Using Eq.(8.20), we can express Eq.(8.26) as

$$\left(\frac{S}{N}\right)_o = 3\left(\frac{k_f^2 S_X}{W^2}\right)\left(\frac{A_c^2}{2\eta B}\right) = 3\left(\frac{k_f^2 S_X}{W^2}\right)\gamma \tag{8.27}$$

Since $\Delta\omega = \left|k_f X(t)\right|_{max} = k_f\,[|X(t)| \le 1]$, Eq.(8.27) can be rewritten as

$$\left(\frac{S}{N}\right)_o = 3\left(\frac{\Delta\omega}{W}\right)^2 S_X \gamma = 3D^2 S_X \gamma \tag{8.28}$$

Equation (8.25) indicates that the output noise power is inversely proportional to the mean carrier power $\left(\frac{A_c^2}{2}\right)$ in *FM*. This effect is called *noise quieting*.

Example 8.1: consider an *FM* broadcast system with parameter Δ*f*=75 kHz and *B*=15 kHz. Assuming $S_X = \frac{1}{2}$, find the output *SNR* and calculate the improvement (in dB) over the baseband system.

Sol.

Substituting the given parameters into Eq.(8.28), we obtain

$$\left(\frac{S}{N}\right)_o = 3\left(\frac{75(10^3)}{15(10^3)}\right)^2\left(\frac{1}{2}\right)\gamma = 37.5\gamma$$

Now, 10 log 37.5=15.7 dB, which indicates that $(S/N)_o$ is about 16 dB better than the baseband system.