7.7 Inverse Trigonometric Functions

Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. They also provide useful antiderivatives and appear frequently in the solutions of differential equations. This section shows how these functions are defined, graphed, and evaluated, how their derivatives are computed, and why they appear as important antiderivatives.

Defining the Inverses

The six basic trigonometric functions are not one-to-one (their values repeat periodically). However we can restrict their domains to intervals on which they are one-to-one. The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse $\sin^{-1} x$ (Figure 7.16). Similar domain restrictions can be applied to all six trigonometric functions.

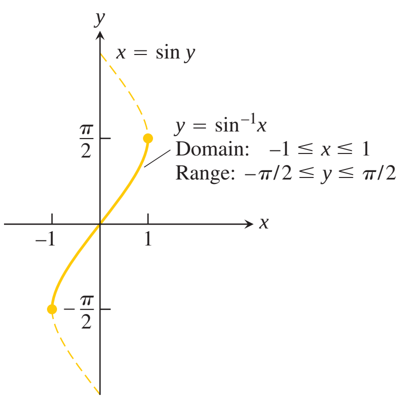
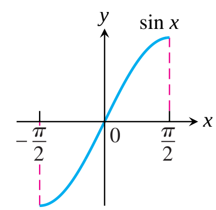
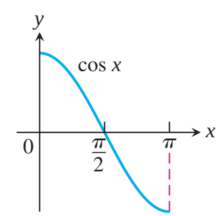
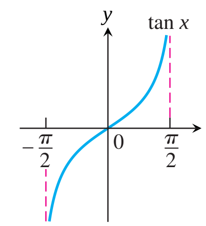
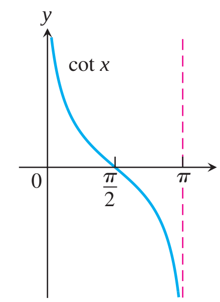
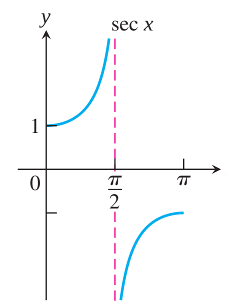
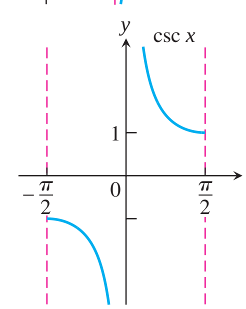


FIGURE 7.16 The graph of $y = \sin^{-1} x$.

Domain restrictions that make the trigonometric functions one-to-one

Function	**Domain**	**Range**	
$\sin x$	$[-\pi/2, \pi/2]$	$[-1, 1]$	
$\cos x$	$[0, \pi]$	$[-1, 1]$	

$\tan x$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$	
$\cot x$	$(0, \pi)$	$(-\infty, \infty)$	
$\sec x$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$	
$\csc x$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$	

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$
\begin{aligned}
y &= \sin^{-1} x \qquad \text{or} \qquad y = \arcsin x \\
y &= \cos^{-1} x \qquad \text{or} \qquad y = \arccos x \\
y &= \tan^{-1} x \qquad \text{or} \qquad y = \arctan x \\
y &= \cot^{-1} x \qquad \text{or} \qquad y = \operatorname{arccot} x \\
y &= \sec^{-1} x \qquad \text{or} \qquad y = \operatorname{arcsec} x \\
y &= \csc^{-1} x \qquad \text{or} \qquad y = \operatorname{arccsc} x
\end{aligned}
$$

These equations are read "y equals the arcsine of x" or "y equals arcsin x" and so on.

CAUTION The -1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

The graphs of the six inverse trigonometric functions are shown in Figure 7.17. We can obtain these graphs by reflecting the graphs of the restricted trigonometric functions through the line $y = x$, as in Section 7.1. We now take a closer look at these functions and their derivatives.

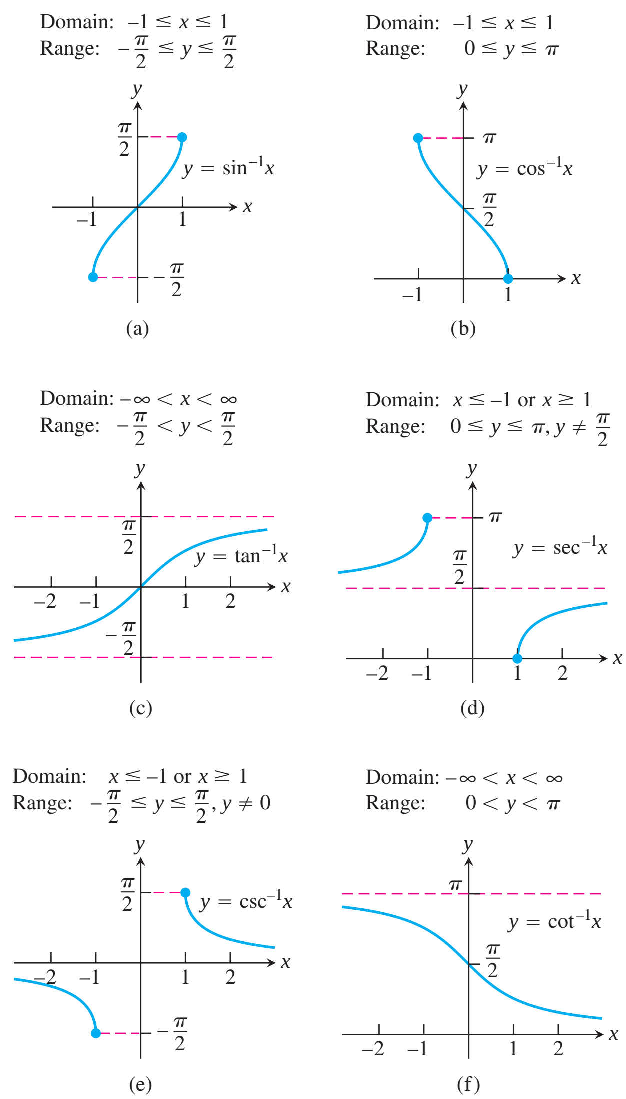


FIGURE 7.17 Graphs of the six basic inverse trigonometric functions.

The Arcsine and Arccosine Functions

The arcsine of x is the angle in $[-\pi/2, \pi/2]$ whose sine is x. The arccosine is an angle in $[0, \pi]$ whose cosine is x.

DEFINITION Arcsine and Arccosine Functions

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

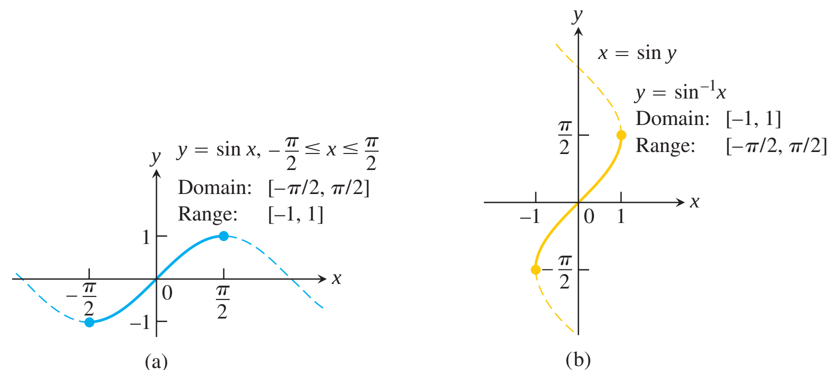
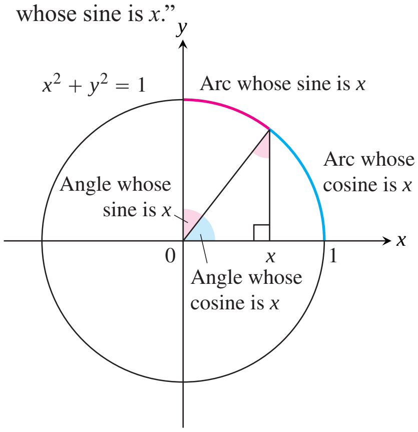


FIGURE 7.18 The graphs of (a) $y = \sin x$, $-\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

The "Arc" in Arc Sine and Arc Cosine

The accompanying figure gives a geometric interpretation of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for radian angles in the first quadrant. For a unit circle, the equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."



The graph of $y = \sin^{-1} x$ (Figure 7.18) is symmetric about the origin (it lies along the graph of $x = \sin y$). The arcsine is therefore an odd function:

$$\sin^{-1}(-x) = -\sin^{-1} x. \tag{1}$$

The graph of $y = \cos^{-1} x$ (Figure 7.19) has no such symmetry.

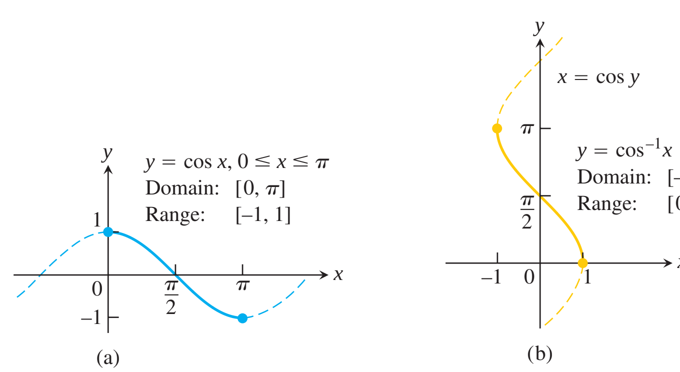
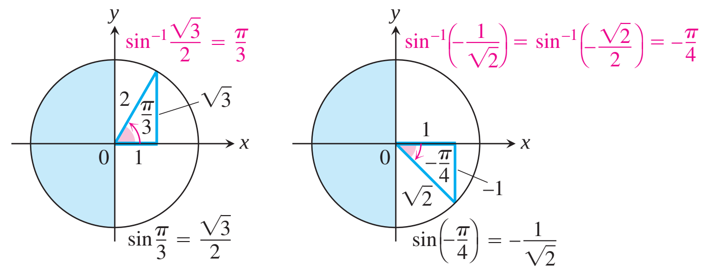


FIGURE 7.19 The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

Known values of $\sin x$ and $\cos x$ can be inverted to find values of $\sin^{-1} x$ and $\cos^{-1} x$.

x	$\sin^{-1} x$
$\sqrt{3}/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$
$-1/2$	$-\pi/6$
$-\sqrt{2}/2$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$

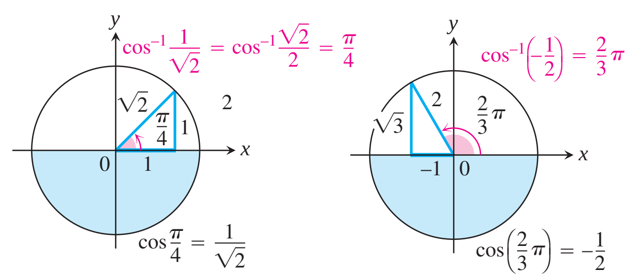
EXAMPLE 1 Common Values of $\sin^{-1} x$



The angles come from the first and fourth quadrants because the range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$. ■

x	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/3$
$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$

EXAMPLE 2 Common Values of $\cos^{-1} x$



The angles come from the first and second quadrants because the range of $\cos^{-1} x$ is $[0, \pi]$. ■

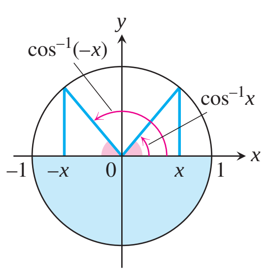


FIGURE 7.20 $\cos^{-1} x$ and $\cos^{-1}(-x)$ are supplementary angles (so their sum is π).

Identities Involving Arcsine and Arccosine

As we can see from Figure 7.20, the arccosine of x satisfies the identity

$$\cos^{-1} x + \cos^{-1}(-x) = \pi, \tag{2}$$

or

$$\cos^{-1}(-x) = \pi - \cos^{-1} x. \tag{3}$$

Also, we can see from the triangle in Figure 7.21 that for $x > 0$,

$$\sin^{-1} x + \cos^{-1} x = \pi/2. \tag{4}$$

Equation (4) holds for the other values of x in $[-1, 1]$ as well, but we cannot conclude this from the triangle in Figure 7.21. It is, however, a consequence of Equations (1) and (3) (Exercise 131).

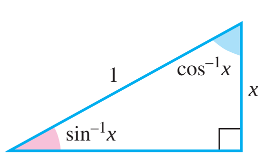


FIGURE 7.21 $\sin^{-1} x$ and $\cos^{-1} x$ are complementary angles (so their sum is $\pi/2$).

Inverses of tan x, cot x, sec x, and csc x

The arctangent of x is an angle whose tangent is x. The arccotangent of x is an angle whose cotangent is x.

DEFINITION Arctangent and Arccotangent Functions

$\boldsymbol{y = \tan^{-1} x}$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$\boldsymbol{y = \cot^{-1} x}$ is the number in $(0, \pi)$ for which $\cot y = x$.

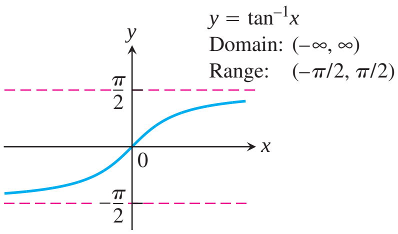


FIGURE 7.22 The graph of $y = \tan^{-1} x$.

We use open intervals to avoid values where the tangent and cotangent are undefined.

The graph of $y = \tan^{-1} x$ is symmetric about the origin because it is a branch of the graph $x = \tan y$ that is symmetric about the origin (Figure 7.22). Algebraically this means that

$$\tan^{-1}(-x) = -\tan^{-1} x;$$

the arctangent is an odd function. The graph of $y = \cot^{-1} x$ has no such symmetry (Figure 7.23).

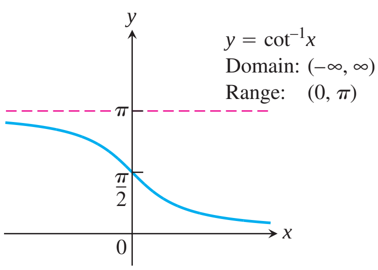


FIGURE 7.23 The graph of $y = \cot^{-1} x$.

The inverses of the restricted forms of $\sec x$ and $\csc x$ are chosen to be the functions graphed in Figures 7.24 and 7.25.

CAUTION There is no general agreement about how to define $\sec^{-1} x$ for negative values of x. We chose angles in the second quadrant between $\pi/2$ and π. This choice makes $\sec^{-1} x = \cos^{-1}(1/x)$. It also makes $\sec^{-1} x$ an increasing function on each interval of its domain. Some tables choose $\sec^{-1} x$ to lie in $[-\pi, -\pi/2)$ for $x < 0$ and some texts choose it to lie in $[\pi, 3\pi/2)$ (Figure 7.26). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $\sec^{-1} x = \cos^{-1}(1/x)$. From this, we can derive the identity

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right) \tag{5}$$

by applying Equation (4).

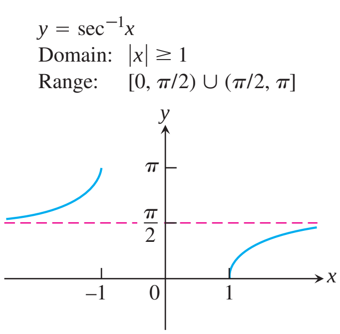


FIGURE 7.24 The graph of $y = \sec^{-1} x$.

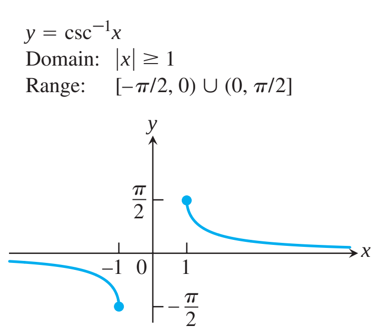


FIGURE 7.25 The graph of $y = \csc^{-1} x$.

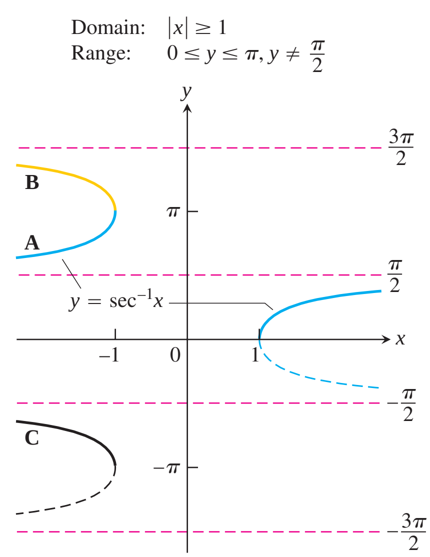
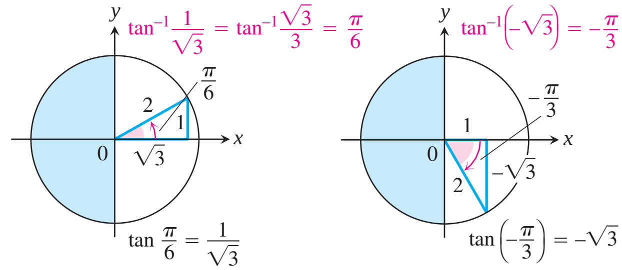


FIGURE 7.26 There are several logical choices for the left-hand branch of $y = \sec^{-1} x$. With choice **A**, $\sec^{-1} x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

EXAMPLE 3 Common Values of $\tan^{-1} x$



The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$. ■

EXAMPLE 4 Find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$ if

$$\alpha = \sin^{-1}\frac{2}{3}.$$

Solution This equation says that $\sin\alpha = 2/3$. We picture α as an angle in a right triangle with opposite side 2 and hypotenuse 3 (Figure 7.27). The length of the remaining side is

$$\sqrt{(3)^2 - (2)^2} = \sqrt{9-4} = \sqrt{5}. \qquad \text{Pythagorean theorem}$$

We add this information to the figure and then read the values we want from the completed triangle:

$$\cos\alpha = \frac{\sqrt{5}}{3}, \quad \tan\alpha = \frac{2}{\sqrt{5}}, \quad \sec\alpha = \frac{3}{\sqrt{5}}, \quad \csc\alpha = \frac{3}{2}, \quad \cot\alpha = \frac{\sqrt{5}}{2}.$$ ■

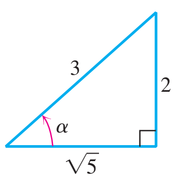


FIGURE 7.27 If $\alpha = \sin^{-1}(2/3)$, then the values of the other basic trigonometric functions of α can be read from this triangle (Example 4).

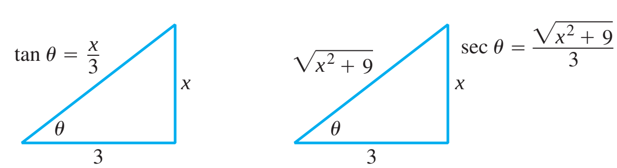
EXAMPLE 5 Find $\sec\left(\tan^{-1}\frac{x}{3}\right)$.

Solution We let $\theta = \tan^{-1}(x/3)$ (to give the angle a name) and picture θ in a right triangle with

$$\tan\theta = \text{opposite}/\text{adjacent} = x/3.$$

The length of the triangle's hypotenuse is

$$\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}.$$



Thus,

$$\begin{aligned}\sec\left(\tan^{-1}\frac{x}{3}\right) &= \sec\theta \\ &= \frac{\sqrt{x^2+9}}{3}. \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}\end{aligned}$$ ■

2.2 Calculating Limits Using the Limit Laws

HISTORICAL ESSAY*

Limits

In Section 2.1 we used graphs and calculators to guess the values of limits. This section presents theorems for calculating limits. The first three let us build on the results of Example 8 in the preceding section to find limits of polynomials, rational functions, and powers. The fourth and fifth prepare for calculations later in the text.

The Limit Laws

The next theorem tells how to calculate limits of functions that are arithmetic combinations of functions whose limits we already know.

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \qquad \text{and} \qquad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $$\lim_{x \to c} (f(x) + g(x)) = L + M$$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $$\lim_{x \to c} (f(x) - g(x)) = L - M$$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $$\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$$

The limit of a product of two functions is the product of their limits.

To learn more about the historical figures and the development of the major elements and topics of calculus, visit **www.aw-bc.com/thomas**.

4. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

It is easy to convince ourselves that the properties in Theorem 1 are true (although these intuitive arguments do not constitute proofs). If x is sufficiently close to c, then $f(x)$ is close to L and $g(x)$ is close to M, from our informal definition of a limit. It is then reasonable that $f(x) + g(x)$ is close to $L + M$; $f(x) - g(x)$ is close to $L - M$; $f(x)g(x)$ is close to LM; $kf(x)$ is close to kL; and that $f(x)/g(x)$ is close to L/M if M is not zero. We prove the Sum Rule in Section 2.3, based on a precise definition of limit. Rules 2–5 are proved in Appendix 2. Rule 6 is proved in more advanced texts.

Here are some examples of how Theorem 1 can be used to find limits of polynomial and rational functions.

EXAMPLE 1 Using the Limit Laws

Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$ (Example 8 in Section 2.1) and the properties of limits to find the following limits.

(a) $\lim_{x \to c}(x^3 + 4x^2 - 3)$ **(b)** $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$ **(c)** $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Solution

(a)
$$\begin{aligned}\lim_{x \to c}(x^3 + 4x^2 - 3) &= \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 && \text{Sum and Difference Rules}\\ &= c^3 + 4c^2 - 3 && \text{Product and Multiple Rules}\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} &= \frac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)} && \text{Quotient Rule}\\ &= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5} && \text{Sum and Difference Rules}\\ &= \frac{c^4 + c^2 - 1}{c^2 + 5} && \text{Power or Product Rule}\end{aligned}$$

(c) $\lim_{x\to -2}\sqrt{4x^2-3} = \sqrt{\lim_{x\to -2}(4x^2-3)}$ Power Rule with $r/s = \frac{1}{2}$

$= \sqrt{\lim_{x\to -2}4x^2 - \lim_{x\to -2}3}$ Difference Rule

$= \sqrt{4(-2)^2-3}$ Product and Multiple Rules

$= \sqrt{16-3}$

$= \sqrt{13}$ ■

Two consequences of Theorem 1 further simplify the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c, merely substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point c *at which the denominator is not zero*, substitute c for x in the formula for the function. (See Examples 1a and 1b.)

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 2 Limit of a Rational Function

$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5} = \frac{0}{6} = 0$$

This result is similar to the second limit in Example 1 with $c = -1$, now done in one step. ■

Identifying Common Factors

It can be shown that if $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x-c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x-c)$ as a common factor.

Eliminating Zero Denominators Algebraically

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c. If this happens, we can find the limit by substitution in the simplified fraction.

EXAMPLE 3 Canceling a Common Factor

Evaluate

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}.$$

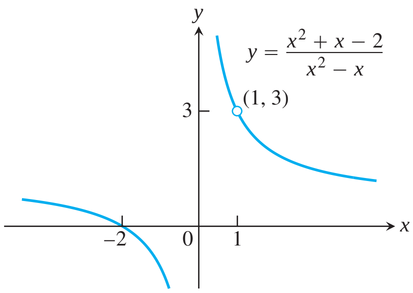
Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling the $(x - 1)$'s gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

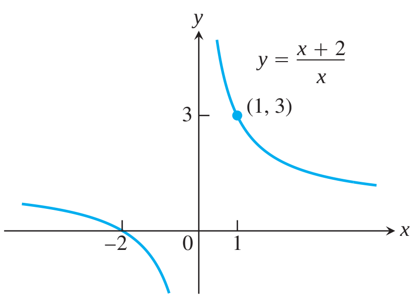
Using the simpler fraction, we find the limit of these values as $x \to 1$ by substitution:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

See Figure 2.8. ■



(a)



(b)

FIGURE 2.8 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 3).

EXAMPLE 4 Creating and Canceling a Common Factor

Evaluate

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

Solution This is the limit we considered in Example 10 of the preceding section. We cannot substitute $x = 0$, and the numerator and denominator have no obvious common factors. We can create a common factor by multiplying both numerator and denominator by the expression $\sqrt{x^2 + 100} + 10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\begin{aligned}
\frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} \\
&= \frac{x^2 + 100 - 100}{x^2\left(\sqrt{x^2 + 100} + 10\right)} \\
&= \frac{x^2}{x^2\left(\sqrt{x^2 + 100} + 10\right)} && \text{Common factor } x^2 \\
&= \frac{1}{\sqrt{x^2 + 100} + 10}. && \text{Cancel } x^2 \text{ for } x \neq 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 100} + 10} \\
&= \frac{1}{\sqrt{0^2 + 100} + 10} && \text{Denominator not 0 at } x = 0\text{; substitute} \\
&= \frac{1}{20} = 0.05.
\end{aligned}$$

This calculation provides the correct answer to the ambiguous computer results in Example 10 of the preceding section. ■

The Sandwich Theorem

The following theorem will enable us to calculate a variety of limits in subsequent chapters. It is called the Sandwich Theorem because it refers to a function f whose values are

EXERCISES 2.2

Limit Calculations

Find the limits in Exercises 1–18.

1. $\lim_{x\to -7}(2x+5)$

2. $\lim_{x\to 12}(10-3x)$

3. $\lim_{x\to 2}(-x^2+5x-2)$

4. $\lim_{x\to -2}(x^3-2x^2+4x+8)$

5. $\lim_{t\to 6} 8(t-5)(t-7)$

6. $\lim_{s\to 2/3} 3s(2s-1)$

7. $\lim_{x\to 2}\dfrac{x+3}{x+6}$

8. $\lim_{x\to 5}\dfrac{4}{x-7}$

9. $\lim_{y\to -5}\dfrac{y^2}{5-y}$

10. $\lim_{y\to 2}\dfrac{y+2}{y^2+5y+6}$

11. $\lim_{x\to -1} 3(2x-1)^2$

12. $\lim_{x\to -4}(x+3)^{1984}$

13. $\lim_{y\to -3}(5-y)^{4/3}$

14. $\lim_{z\to 0}(2z-8)^{1/3}$

15. $\lim_{h\to 0}\dfrac{3}{\sqrt{3h+1}+1}$

16. $\lim_{h\to 0}\dfrac{5}{\sqrt{5h+4}+2}$

17. $\lim_{h\to 0}\dfrac{\sqrt{3h+1}-1}{h}$

18. $\lim_{h\to 0}\dfrac{\sqrt{5h+4}-2}{h}$

Find the limits in Exercises 19–36.

19. $\lim_{x\to 5}\dfrac{x-5}{x^2-25}$

20. $\lim_{x\to -3}\dfrac{x+3}{x^2+4x+3}$

21. $\lim_{x\to -5}\dfrac{x^2+3x-10}{x+5}$

22. $\lim_{x\to 2}\dfrac{x^2-7x+10}{x-2}$

23. $\lim_{t\to 1}\dfrac{t^2+t-2}{t^2-1}$

24. $\lim_{t\to -1}\dfrac{t^2+3t+2}{t^2-t-2}$

25. $\lim_{x\to -2}\dfrac{-2x-4}{x^3+2x^2}$

26. $\lim_{y\to 0}\dfrac{5y^3+8y^2}{3y^4-16y^2}$

27. $\lim_{u\to 1}\dfrac{u^4-1}{u^3-1}$

28. $\lim_{v\to 2}\dfrac{v^3-8}{v^4-16}$

29. $\lim_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}$

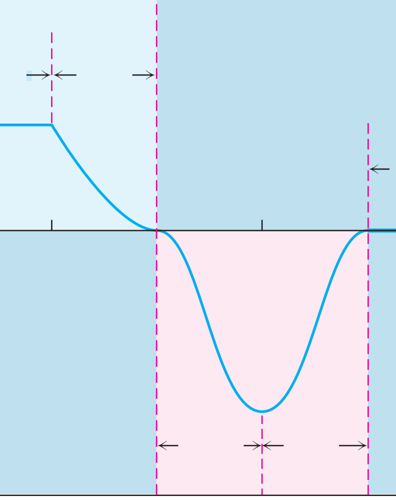
30. $\lim_{x\to 4}\dfrac{4x-x^2}{2-\sqrt{x}}$

31. $\lim_{x\to 1}\dfrac{x-1}{\sqrt{x+3}-2}$

32. $\lim_{x\to -1}\dfrac{\sqrt{x^2+8}-3}{x+1}$

33. $\lim_{x\to 2}\dfrac{\sqrt{x^2+12}-4}{x-2}$

34. $\lim_{x\to -2}\dfrac{x+2}{\sqrt{x^2+5}-3}$



Chapter 3

Differentiation

OVERVIEW In Chapter 2, we defined the slope of a curve at a point as the limit of secant slopes. This limit, called a derivative, measures the rate at which a function changes, and it is one of the most important ideas in calculus. Derivatives are used to calculate velocity and acceleration, to estimate the rate of spread of a disease, to set levels of production so as to maximize efficiency, to find the best dimensions of a cylindrical can, to find the age of a prehistoric artifact, and for many other applications. In this chapter, we develop techniques to calculate derivatives easily and learn how to use derivatives to approximate complicated functions.

3.1 The Derivative as a Function

Historical Essay

The Derivative

At the end of Chapter 2, we defined the slope of a curve $y = f(x)$ at the point where $x = x_0$ to be

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We called this limit, when it existed, the derivative of f at x_0. We now investigate the derivative as a *function* derived from f by considering the limit at each point of the domain of f.

DEFINITION Derivative Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

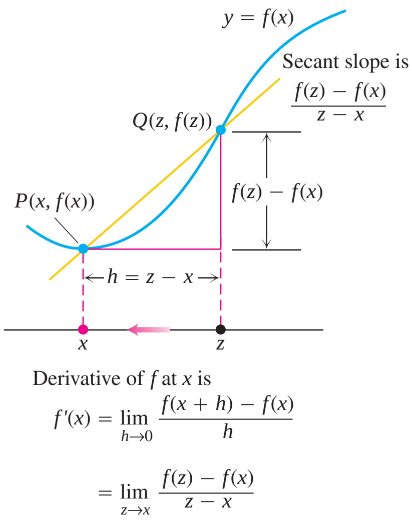


FIGURE 3.1 The way we write the difference quotient for the derivative of a function f depends on how we label the points involved.

We use the notation $f(x)$ rather than simply f in the definition to emphasize the independent variable x, which we are differentiating with respect to. The domain of f' is the set of points in the domain of f for which the limit exists, and the domain may be the same or smaller than the domain of f. If f' exists at a particular x, we say that f is **differentiable (has a derivative)** at x. If f' exists at every point in the domain of f, we call f **differentiable**.

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x. Therefore, an equivalent definition of the derivative is as follows (see Figure 3.1).

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

as another way to denote the derivative $f'(x)$. Examples 2 and 3 of Section 2.7 illustrate the differentiation process for the functions $y = mx + b$ and $y = 1/x$. Example 2 shows that

$$\frac{d}{dx}(mx + b) = m.$$

For instance,

$$\frac{d}{dx}\left(\frac{3}{2}x - 4\right) = \frac{3}{2}.$$

In Example 3, we see that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Here are two more examples.

EXAMPLE 1 Applying the Definition

Differentiate $f(x) = \dfrac{x}{x - 1}$.

Solution Here we have $f(x) = \dfrac{x}{x - 1}$

and

$$f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}.
\end{aligned}$$

■

EXAMPLE 2 Derivative of the Square Root Function

(a) Find the derivative of $y = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

You will often need to know the derivative of $\sqrt{x}$ for $x > 0$:

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Solution

(a) We use the equivalent form to calculate f':

$$\begin{aligned}
f'(x) &= \lim_{z\to x} \frac{f(z)-f(x)}{z-x} \\
&= \lim_{z\to x} \frac{\sqrt{z}-\sqrt{x}}{z-x} \\
&= \lim_{z\to x} \frac{\sqrt{z}-\sqrt{x}}{\left(\sqrt{z}-\sqrt{x}\right)\left(\sqrt{z}+\sqrt{x}\right)} \\
&= \lim_{z\to x} \frac{1}{\sqrt{z}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.
\end{aligned}$$

(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point (4, 2) with slope 1/4 (Figure 3.2):

$$y = 2 + \frac{1}{4}(x-4)$$

$$y = \frac{1}{4}x + 1.$$

■

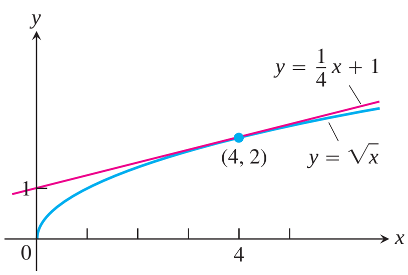


FIGURE 3.2 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

We consider the derivative of $y = \sqrt{x}$ when $x = 0$ in Example 6.