Logic & Propositional Calculus

Introduction

- Propositional calculus (or logic) is the study of the logical relationship between objects called propositions and forms the basis of all mathematical reasoning.

- **Definition:** A proposition is a statement that is either true or false, but not both (we usually denote a proposition by letters; p, q, r, s, . . .).

Introduction

- **Definition:** The value of a proposition is called its truth value; denoted by T or 1 if it is true and F or 0 if it is false.

- *Truth Table:*

p
0
1

Examples

- Example (Propositions)
  -Today is Monday.
  -The derivative of sin x is cos x.
  -Every even number has at least two factors.

- Example (Not Propositions)
  -When is the pretest?
  -Do your homework.

Basic Logical Operations

- Connectives are used to create a compound proposition from two or more other propositions.

  -Negation (denoted $\neg$ or !)

  -And (denoted $\wedge$) or Logical Conjunction

  -Or (denoted $\vee$) or Logical Disjunction

  -Exclusive Or (XOR, denoted $\oplus$)

  -Implication (denoted $\rightarrow$)

  -Biconditional; "if and only if" (denoted $\leftrightarrow$)

Basic Logical Operations (Cont.)

Formal Name	Nickname	Arity	Symbol
Negation operator	NOT	Unary	$\neg$
Conjunction operator	AND	Binary	$\wedge$
Disjunction operator	OR	Binary	$\vee$
Exclusive-OR operator	XOR	Binary	$\oplus$
Implication operator	IMPLIES	Binary	$\rightarrow$
Biconditional operator	IFF	Binary	$\leftrightarrow$

Boolean Operation Summary

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
F	F	T	F	F	F	T	T
F	T	T	F	T	T	T	F
T	F	F	F	T	T	F	F
T	T	F	T	T	F	T	T

Example (1)

- Construct the Truth Table for the following compound proposition.

$$((p \wedge q) \vee \neg q).$$

p	q	$p \wedge q$	$\neg q$	$((p \wedge q) \vee \neg q)$
0	0	0	1	1
0	1	0	0	0
1	0	0	1	1
1	1	1	0	1

Example (2)

- Construct the Truth Table for the following compound proposition.

$$q \wedge (\neg r \rightarrow p).$$

p	q	r	$\neg r$	$\neg r \rightarrow p$	$q \wedge (\neg r \rightarrow p)$
0	0	0	1	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	0	1	1
1	0	0	1	1	0
1	0	1	0	1	0
1	1	0	1	1	1
1	1	1	0	1	1

Example (3)

- Find if the following two compound propositions are equivalence?

$$p \vee (q \wedge r) \qquad\qquad (p \vee q) \wedge r$$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$(p \vee q) \wedge r$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	1	0
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	0	1	1	1
1	1	0	0	1	1	0
1	1	1	1	1	1	1

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Homework

- *Construct the truth table for the following logical expression?*

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

Limitations of Truth Tables

- It just requires some brute force calculation.
- If we have 129 variables, truth table has 2^{129} lines. That number is larger than the number of atoms in the known universe.
- We don't need any insight into *why the proposition you're* working on is true (or false, or sometimes true).
- However, a big advantage of the Boolean algebra, is that they give much better insight into *why a theorem is true.*

Tautologies & Contradictions

- A tautology is a compound proposition which is true no matter what the truth values of its simple components.
- A contradiction is a compound proposition which is false no matter what the truth values of its simple components.

Tautologies & Contradictions (Cont.)

- Show that (p ∧ q) ∨ ¬(p ∧ q) is a tautology.

p	q	$p \wedge q$	$\overline{p \wedge q}$	$(p \wedge q) \vee (\overline{p \wedge q})$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Logical Equivalence

- Two propositions are said to be logically equivalent if they have identical truth values for every set of truth values of their components.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Logical Equivalence (Cont.)

- ***Example***

p	q	$p \to q$	$q \to p$	$(p \to q) \land (q \to p)$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

Algebra of Propositions

Idempotent laws

(1a) $p \vee p \equiv p$ (1b) $p \wedge p \equiv p$

Associative laws

(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$ (2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Commutative laws

(3a) $p \vee q \equiv q \vee p$ (3b) $p \wedge q \equiv q \wedge p$

Distributive laws

(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Identity laws

(5a) $p \vee \mathrm{F} \equiv p$ (5b) $p \wedge \mathrm{T} \equiv p$

(6a) $p \vee \mathrm{T} \equiv \mathrm{T}$ (6b) $p \wedge \mathrm{F} \equiv \mathrm{F}$

Algebra of Propositions (Cont.)

Complement laws	
(7*a*) $p \vee \neg p \equiv \mathrm{T}$	(7*b*) $p \wedge \neg p \equiv \mathrm{F}$
(8*a*) $\neg \mathrm{T} \equiv \mathrm{F}$	(8*b*) $\neg \mathrm{F} \equiv \mathrm{T}$
Involution law	
(9) $\neg\neg p \equiv p$	
DeMorgan's laws	
(10*a*) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10*b*) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Example (1)

- Simplify the following logical expression

$$(p \vee q) \wedge \neg(\neg p \wedge q)$$

	Reasons
$(p \vee q) \wedge \neg(\neg p \wedge q)$	
$\Leftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q)$	DeMorgan's Law
$\Leftrightarrow (p \vee q) \wedge (p \vee \neg q)$	Law of Double Negation
$\Leftrightarrow (p \vee (q \wedge \neg q)$	Distributive Law of $\vee$ over $\wedge$
$\Leftrightarrow p \vee F_0$	Inverse Law
$\Leftrightarrow p$	Identity Law

Consequently, we see that

$$(p \vee q) \wedge \neg(\neg p \wedge q) \Leftrightarrow p,$$

Prove Equivelance

Prove $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$.

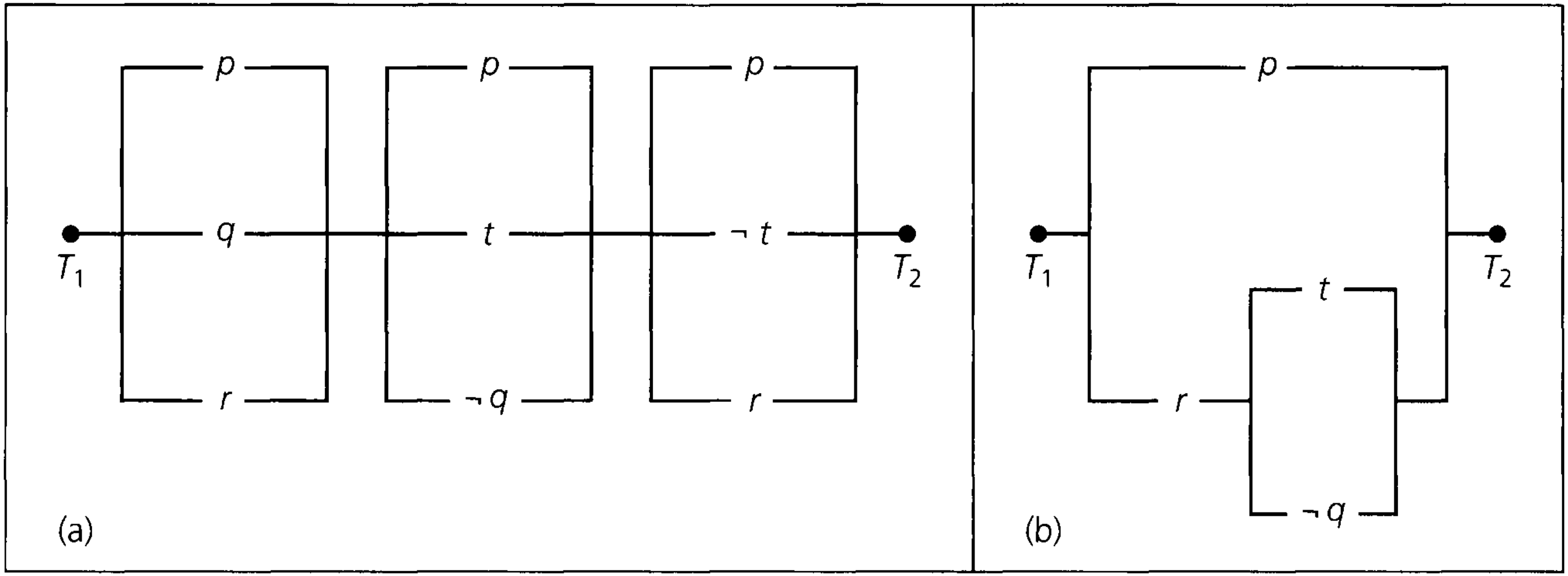
$\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg(\neg p \land q)$ DeMorgan

$\equiv \neg p \land (p \lor \neg q)$ DeMorgan

$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$ Distributivity

$\equiv \neg p \land \neg q$

Applications



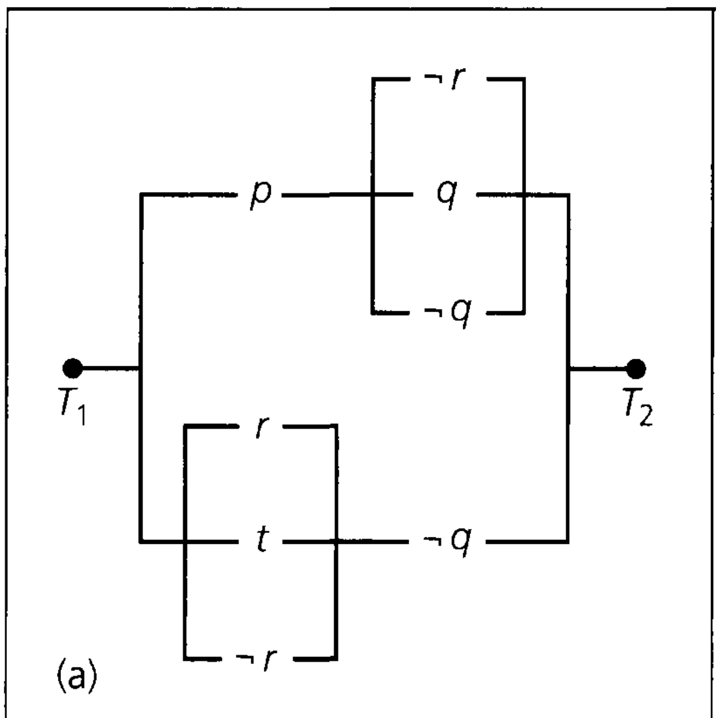
This network is represented by the statement $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$. Using the laws of logic, we may simplify this statement as follows.

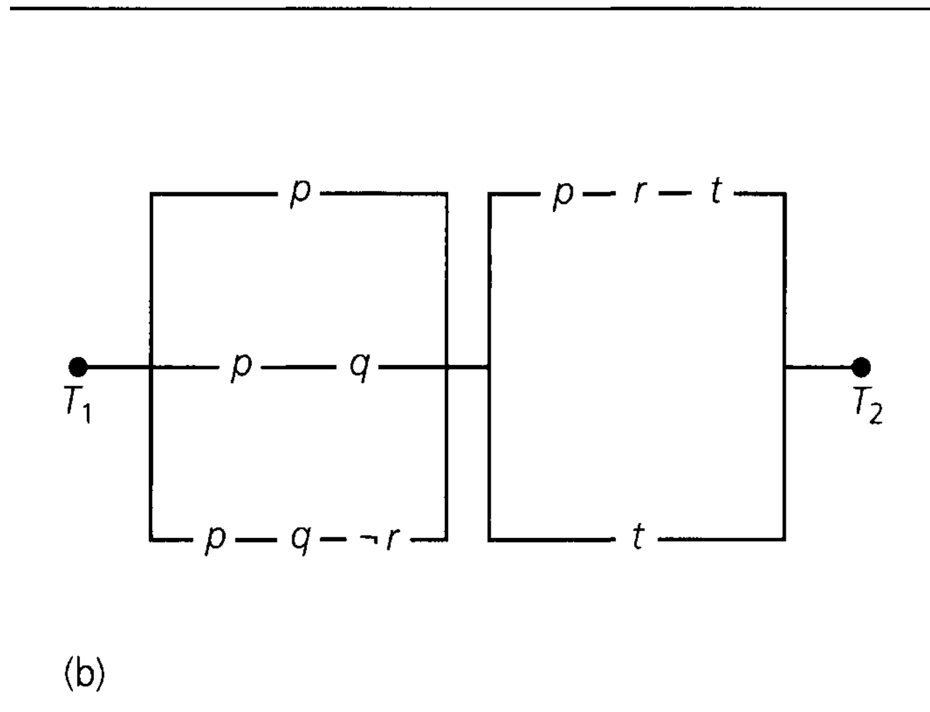
Applications (Cont.)

	Reasons
$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$	
$\Leftrightarrow p \vee [(q \vee r) \wedge (t \vee \neg q) \wedge (\neg t \vee r)]$	Distributive Law of $\vee$ over $\wedge$
$\Leftrightarrow p \vee [(q \vee r) \wedge (\neg t \vee r) \wedge (t \vee \neg q)]$	Commutative Law of $\wedge$
$\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (t \vee \neg q)]$	Distributive Law of $\vee$ over $\wedge$
$\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (\neg\neg t \vee \neg q)]$	Law of Double Negation
$\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge \neg(\neg t \wedge q)]$	DeMorgan's Law
$\Leftrightarrow p \vee [\neg(\neg t \wedge q) \wedge ((\neg t \wedge q) \vee r)]$	Commutative Law of $\wedge$ (twice)
$\Leftrightarrow p \vee [(\neg(\neg t \wedge q) \wedge (\neg t \wedge q)) \vee (\neg(\neg t \wedge q) \wedge r)]$	Distributive Law of $\wedge$ over $\vee$
$\Leftrightarrow p \vee [F_0 \vee (\neg(\neg t \wedge q) \wedge r)]$	$\neg s \wedge s \Leftrightarrow F_0$, for any statement s
$\Leftrightarrow p \vee [(\neg(\neg t \wedge q)) \wedge r]$	F_0 is the identity for $\vee$
$\Leftrightarrow p \vee [r \wedge \neg(\neg t \wedge q)]$	Commutative Law of $\wedge$
$\Leftrightarrow p \vee [r \wedge (t \vee \neg q)]$	DeMorgan's Law and the Law of Double Negation

Homework

- *Try to find the logical expression for the following Circuits. Is there any relation between the two?*





The date for

the next exam is

January 3, 2012;

Tuesday.

Be ready for the exam on the next lecture

References

1. John O'Donnell, Cordelia Hall, and Rex Page, "Discrete Mathematics Using a Computer," 2nd edition, Springer-Verlag, 2006.
2. Seymour Lipschutz, and Marc Lipson, "Schaum's Outlines: Discrete Mathematics," 3rd edition, McGraw-Hill, 2007.
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4. Rowan Garnier, and John Taylor "Discrete Mathematics for New Technology," 2nd edition, Institute of Physics Publishing, 2002.
5. Vladlen Koltun, "Discrete Structures Lecture Notes," 2006.
6. Miguel A. Lerma, "Notes on Discrete Mathematics" Northwestern University, 2005.

Thank You for Listening.