



Functions

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Introduction

- **Definition 1:** A function or mapping f from a set A to a set B , denoted $f: A \rightarrow B$, is a correspondence in which to each element x of A corresponds exactly one element $y = f(x)$ of B .
- **Definition 2.** Let A and B be sets. A function f with type $A \rightarrow B$ is a relation with domain A and codomain B , such that $\forall x \in A. \forall y_1 \in B. \forall y_2 \in B. ((x, y_1) \in f \wedge (x, y_2) \in f) \rightarrow y_1 = y_2$.

Example

Figure 3-1 defines a function f from $A = \{a, b, c, d\}$ into $B = \{r, s, t, u\}$ in the obvious way. Here

$$f(a) = s, \quad f(b) = u, \quad f(c) = r, \quad f(d) = s$$

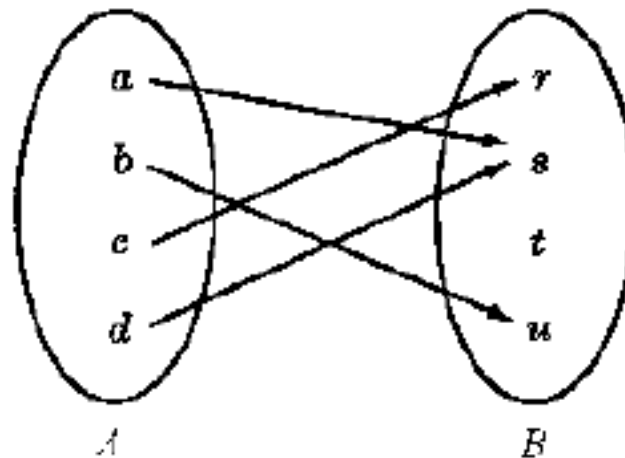


Fig. 3-1

Examples on Functions

- The function **Floor** accepts any real number as input and outputs the integer formed by truncating the fractional part of the number input. For example, $\text{Floor}(3.14159) = \lfloor 3.14159 \rfloor = 3$.
- The function **Ceiling** accepts any real number as input and outputs the smallest integer greater than or equal to the number input. For example, $\text{Ceiling}(3.14159) = \lceil 3.14159 \rceil = 4$. This function is also referred to as the greatest integer function.

Domain and Range

- The **domain** of a function is the set of all things that may be input to produce some output.
- The **range** of a function is the set of all things that are output. Once one knows the domain of a function, one can determine the range by applying the function to each element of the domain.
- The **codomain** of a function is the set of all values that are potential outputs.

Identity Function

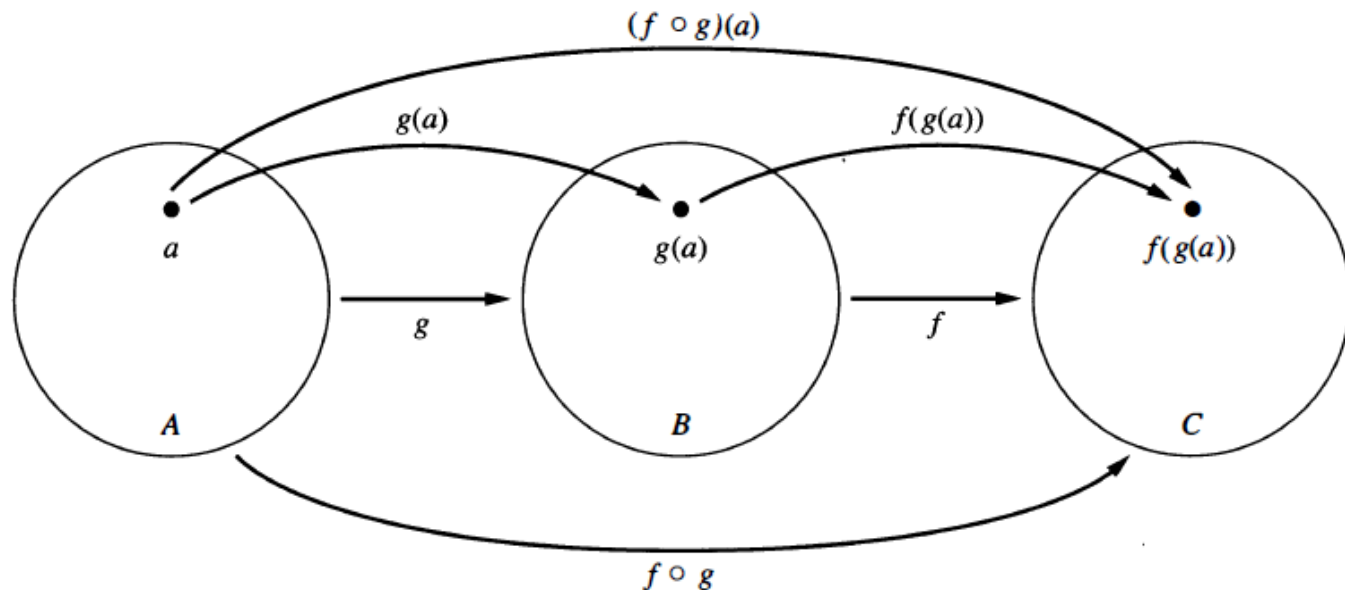
■ **Definition:** Given a set A , the function $1_A : A \rightarrow A$ defined by $1_A(x) = x$ for every x in A is called the *identity function* for A .

■ **Example**

Let $A = \{1, 2, 3\}$ and f is the identity function, then $f(1) = 1$, $f(2) = 2$, and $f(3) = 3$.

Composition of a Function

- Let g be a function from the set A to the set B and let f be a function from the set B to the set C . The composition of the functions f and g , denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.



Example

- Let g be the function from the set $\{a,b,c\}$ to itself such that $g=\{(a,b), (b,c), (c,a)\}$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f=\{(a,3), (b,2),(c,1)\}$. What is the composition of f and g , and what is the composition of g and f ?

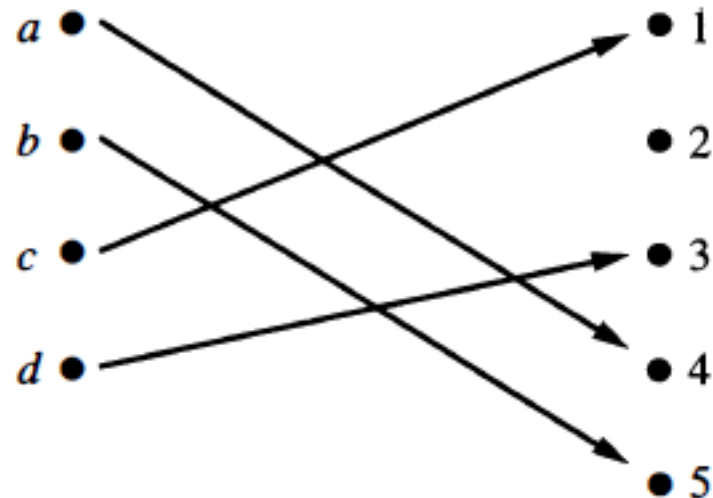
- **Solution:**

The composition $f \circ g$ is defined by $(f \circ g)(a)=f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$.

Note that $g \circ f$ is not defined.

ONE-TO-ONE Functions

- Some functions never assign the same value to two different domain elements. These functions are said to be **ONE-TO-ONE**.
- The function f from $\{a,b,c,d\}$ to $\{1,2,3,4,5\}$ where $f = \{(a,4), (b,5), (c,1), (d,3)\}$ is one-to-one because f takes on different values at the four elements of its domain.



Examples on 1-1 Function

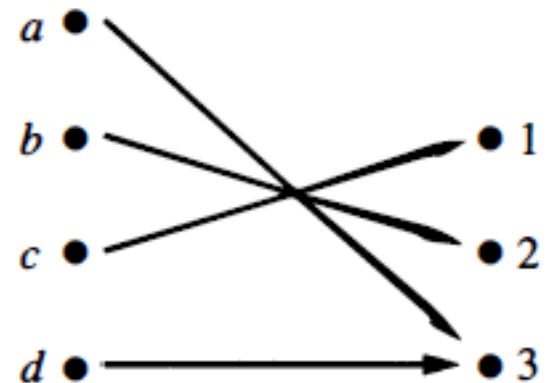
- The function $f(x) = x^2$ is not one-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 \neq -1$
- The function $f(x) = x^2$ with its domain restricted to \mathbb{Z}_+ (positive int.) is one-to-one.
- The function $f(x) = x + 1$ is a one-to-one function.

Onto Function

- A function f from A to B is called **onto** if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

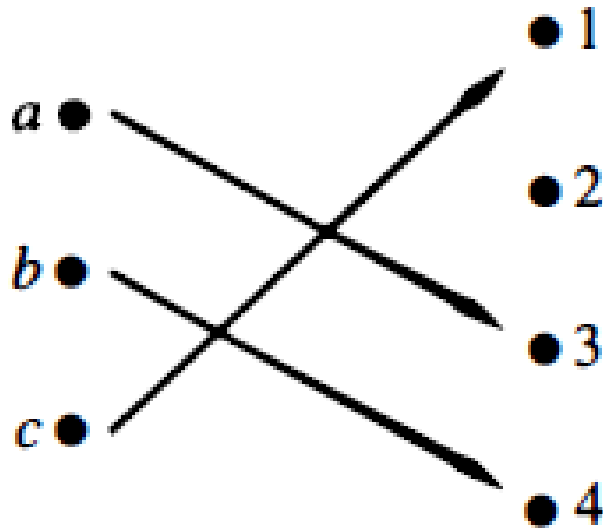
- **Example:** Because all three elements

of the codomain are images of elements in the domain, we see that f is onto.

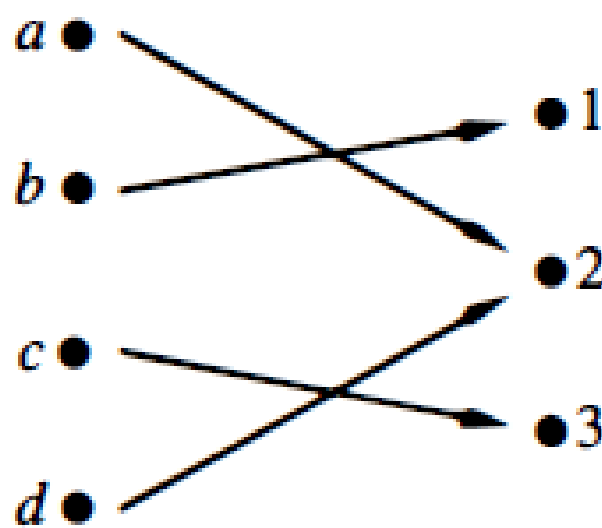


Examples on 1-1 & onto Function

(a) One-to-one,
not onto

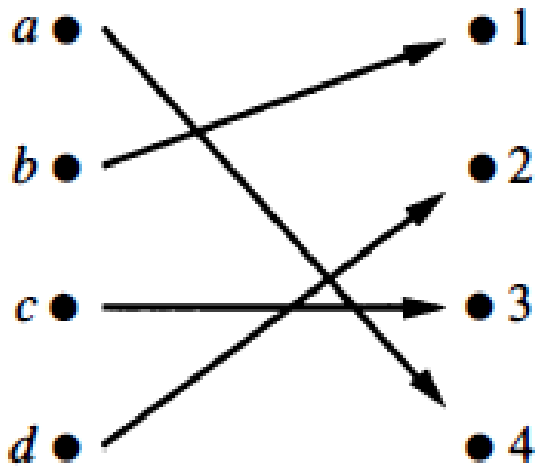


(b) Onto,
not one-to-one

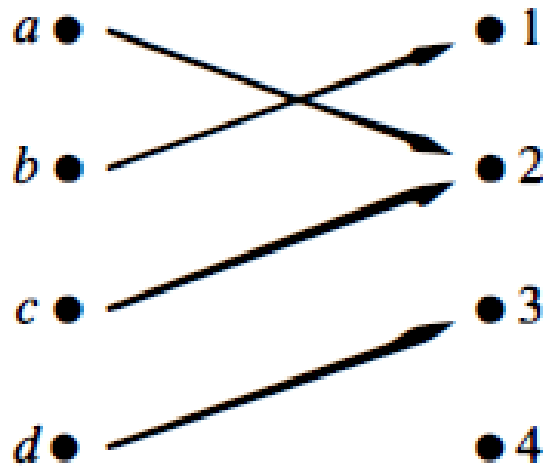


Examples on 1-1 & onto Function

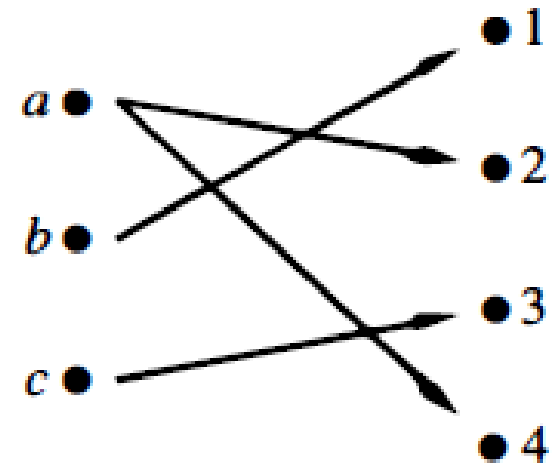
(c) One-to-one,
and onto



(d) Neither one-to-one
nor onto



(e) Not a function



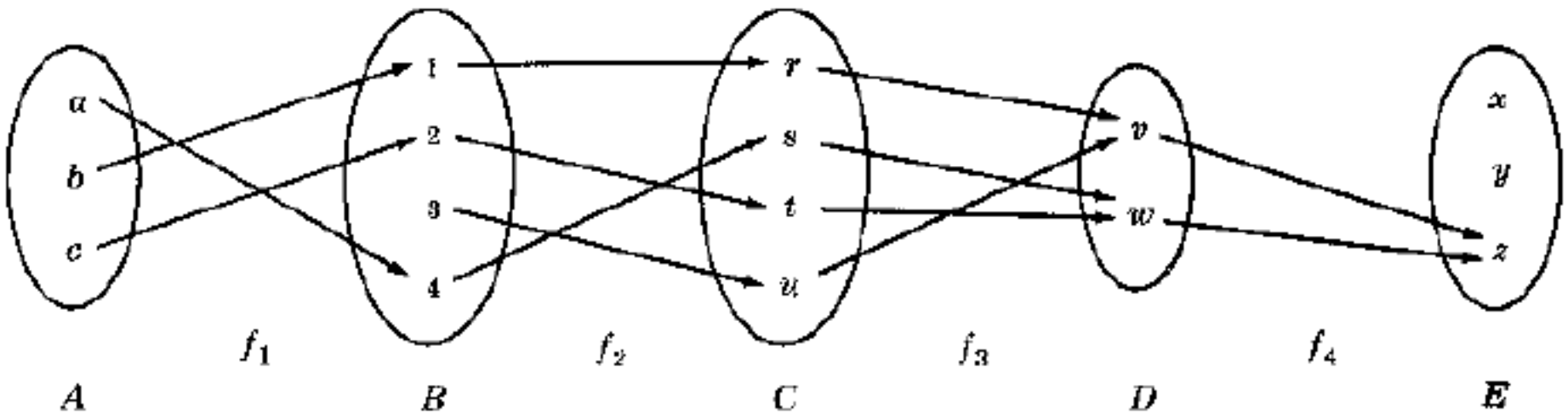
Examples on 1-1 & onto Function

■ f_1 is 1-1 but not onto

■ f_3 is not 1-1 but onto

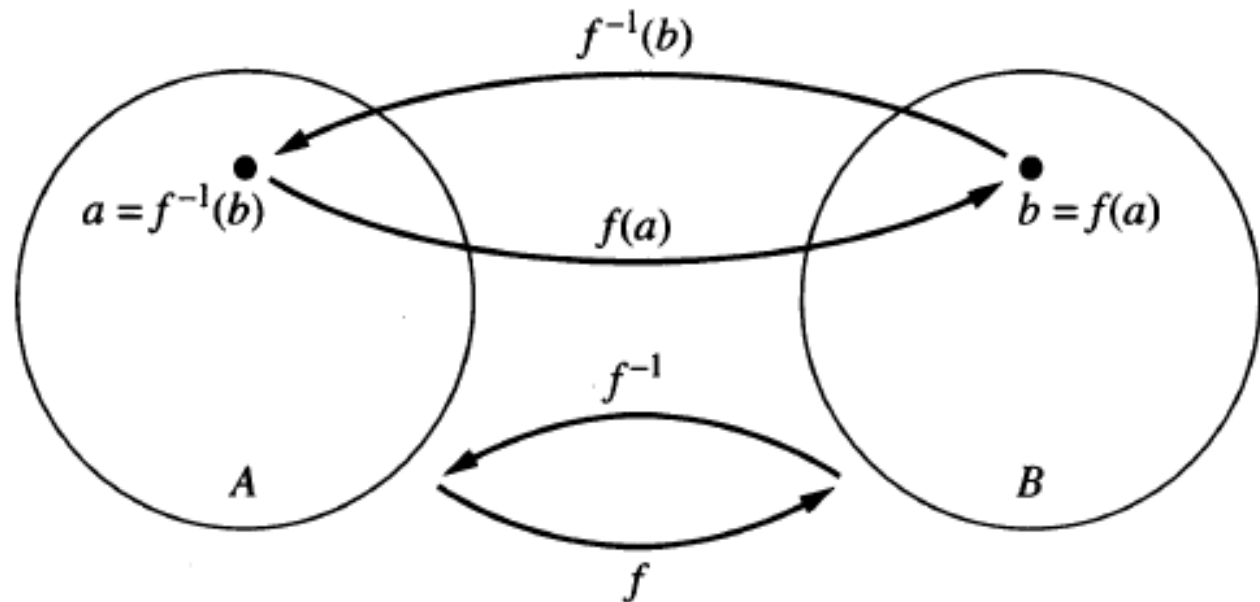
f_2 is both.

f_4 is neither 1-1 nor onto



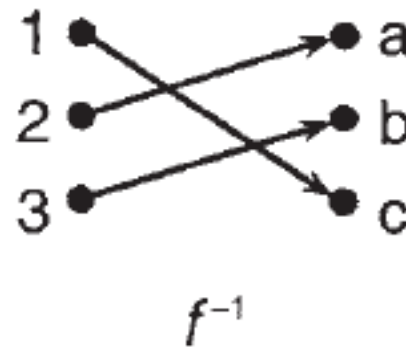
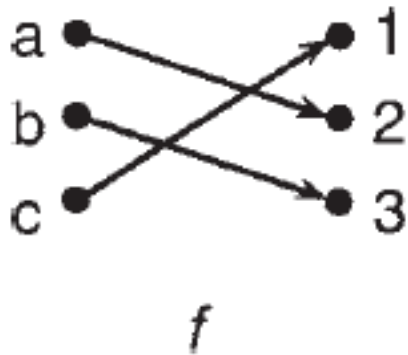
Invertible Functions

- Let f be a one-to-one correspondence from the set A to the set B . The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1} . Hence, $f^{-1}(b) = a$ when $f(a) = b$.



Invertible Functions (Cont.)

- A function $f: A \rightarrow B$ is invertible if and only if f is both 1-1 and onto.



Popular Functions in Computer Sci.

- Floor & Ceiling Functions.

- Integer & Absolute Value Functions.

$$\text{INT}(3.14)=3 \quad \text{INT}(-8.5)=-8 \quad \text{INT}(7)=7$$

$$\text{ABS}(7)=7 \quad \text{ABS}(-15)=15 \quad \text{ABS}(-3.3)=3.3$$

- Remainder Func. & Modular Func.

$$25(\text{mod } 7)=4 \quad -26(\text{mod } 7) = 7 - 5 = 2$$

- Exponential Function a^b

- Logarithm Function $\log_b x$

Sequence

- Let a be a whole number and $X = \{a, a+1, a+2, \dots\}$. A function s with domain X or a subset of X is called a sequence. Let $n \in X$. Then $s(n)$ is called a **term** of the sequence, denoted by s_n . The various terms of the sequence can be listed as $s_a, s_{a+1}, s_{a+2}, \dots$ in increasing order of subscripts.
- In particular, let $X = \mathbb{N}$. Then the terms of the sequence are:

$$s_1, s_2, s_3, \dots, s_n, \dots$$

↑
general term

Sequences (Cont.)

- **Example** Consider the sequence $\{s_n\}$, where $s_n = 2n - 1$. The various terms of the sequence are 1,3,5,7,... Formally, the sequence is the function $s : \mathbb{N} \rightarrow \mathbb{N}$ defined by $s(n) = 2n-1$.
- A sequence is **finite** if its domain is finite; otherwise, it is **infinite**.

Summation Notation

- Sums such as $a + a_k + \dots + a_m$ can be written in a compact form using the summation symbol Σ , which denotes the word sum.
- The summation notation was introduced in 1772 by the brilliant French mathematician Joseph Louis Lagrange.

$$\sum_{i=k}^{i=m} a_i = a_k + a_{k+1} + \dots + a_m$$

$$\sum_{i=k}^{i=m} a_i = \sum_{i=k}^m a_i = \sum_k^m a_i$$

Summation Notation (Cont.)

$$\sum_{i=1}^6 i = 1 + 2 + 3 + 4 + 5 + 6 = 21$$

$$\sum_{i=-1}^2 i(i-1) = (-1)(-1-1) + 0(0-1) + 1(1-1) + 2(2-1) = 4$$

$$\sum_{i=-2}^3 i^2 = (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 = 19$$

Summation's Laws

$$\sum_{i=1}^n c = nc$$

$$\sum_{i=1}^n (ca_i) = c \left(\sum_{i=1}^n a_i \right)$$

$$\sum_{i=1}^n (a_i + b_i) = \left(\sum_{i=1}^n a_i \right) + \left(\sum_{i=1}^n b_i \right)$$

Summation Notation (Cont.)

$$1 + 4 + 9 + 16 + \dots + n^2 = \sum_{i=1}^n i^2.$$

$$1 + 2 + 4 + 8 + \dots + 2^n = \sum_{i=0}^n 2^i.$$

$$7 \cdot 2^3 + 7 \cdot 2^4 + 7 \cdot 2^5 + 7 \cdot 2^6 + \dots + 7 \cdot 2^{n-1} = \sum_{i=3}^{n-1} 7 \cdot 2^i.$$

$$1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \dots + (-1)^{n+1} \frac{1}{n!} = \sum_{i=1}^n \frac{(-1)^{i+1}}{i!}$$

Summation Notation (Cont.)

$$\begin{aligned}\sum_{j=-1}^2 [(5j)^3 - 2j] &= \left(\sum_{j=-1}^2 (5j)^3 - 2 \sum_{j=-1}^2 j \right) \\ &= 125 \left(\sum_{j=-1}^2 j^3 \right) - 2 \sum_{j=-1}^2 j \\ &= 125[(-1)^3 + 0^3 + 1^3 + 2^3] - 2(-1 + 0 + 1 + 2) \\ &= 996\end{aligned}$$

Summation Notation (Cont.)

$$\begin{aligned}\sum_{i=-1}^1 \sum_{j=0}^2 (2i + 3j) &= \sum_{i=-1}^1 \left[\sum_{j=0}^2 (2i + 3j) \right] \\ &= \sum_{i=-1}^1 [(2i + 3 \cdot 0) + (2i + 3 \cdot 1) + (2i + 3 \cdot 2)] \\ &= \sum_{i=-1}^1 (6i + 9) \\ &= [6 \cdot (-1) + 9] + (6 \cdot 0 + 9) + (6 \cdot 1 + 9) \\ &= 27\end{aligned}$$

Recursively Defined Functions

- When a function F is defined by a formula, we can find the value of F at any element of its domain without knowing its value at any other element of its domain.
- Example:

Consider the function $F : \mathbb{N} \rightarrow \mathbb{N}$ defined by the rule

$F(n) = 3n + 2$. We can compute directly that:

$$F(100) = 3 * 100 + 2 = 302 \text{ or that}$$

$$F(3112) = 3 * 3112 + 2 = 9338.$$

Recursively Defined Functions (Cont.)

- The Functions, however, are not necessarily defined in such a straightforward manner. Consider the function $G : \mathbb{N} \rightarrow \mathbb{N}$ defined as $G(0) = 2$ and, for $n > 0$, $G(n) = G(n - 1) + 3$.
- Then, $G(1) = G(0) + 3 = 2 + 3 = 5$. The following computation shows how $G(5)$ would be determined:

$$\begin{aligned}G(5) &= G(4) + 3 \\ &= G(3) + 3 + 3 \\ &= G(2) + 3 + 3 + 3 \\ &= G(1) + 3 + 3 + 3 + 3 \\ &= G(0) + 3 + 3 + 3 + 3 + 3 \\ &= 2 + 3 \cdot 5 \\ &= 17\end{aligned}$$

Recursively Defined Functions (Cont.)

- How many $G(520)$ will call itself?
- Do you know that the last two functions F and G are actually the same; that is, $F(n) = G(n)$ for every $n \in \mathbb{N}$.

Recursively Defined Functions (Cont.)

- How to compute number four factorial func. (4!)?

$$n! = n * (n - 1)!$$

$$(1) \quad 4! = 4 \cdot 3!$$

$$(2) \quad \quad \quad 3! = 3 \cdot 2!$$

$$(3) \quad \quad \quad \quad 2! = 2 \cdot 1!$$

$$(4) \quad \quad \quad \quad \quad 1! = 1 \cdot 0!$$

$$(5) \quad \quad \quad \quad \quad \quad 0! = 1$$

$$(6) \quad \quad \quad \quad \quad \quad 1! = 1 \cdot 1 = 1$$

$$(7) \quad \quad \quad \quad \quad \quad 2! = 2 \cdot 1 = 2$$

$$(8) \quad \quad \quad \quad \quad \quad 3! = 3 \cdot 2 = 6$$

$$(9) \quad 4! = 4 \cdot 6 = 24$$

Recursively Defined Functions (Cont.)

- Examples on functions that can be converted to recursive function

No.	Normal Function	Recursive Function
1	$F(n)=3^n$	$F(0)=1$ $F(n)=3 * F(n - 1)$ for $n \geq 1$
2	$SUM(k)=a_1+a_2+\dots+a_k$ where $1 \leq k \leq n$	$S(1)=a_1$ $S(k)=S(k - 1)+a_k$ for $k > 1$
3	$a+ ac+ ac^2 +ac^3 +\dots+ ac^{n-1}$	$gs(0)=a$ $gs(k)=gs(k - 1)+ac^k$ for $k \geq 1$
4	$n *(n - 1) *(n - 2)*\dots*3*2*1$	$n! = n * (n - 1)!$

Recursively Defined Functions (Cont.)

- **Faibonacci Sequence**: each succeeding term is the sum of the two preceding terms.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

$$F(0) = 1$$


$$F(1) = 1$$

$$F(2) = F(1) + F(0) = 1 + 1 = 2$$

$$F(3) = F(2) + F(1) = 2 + 1 = 3$$

$$F(4) = F(3) + F(2) = 3 + 2 = 5$$

- (a) If $n = 0$ or $n = 1$, then $F_n = n$.
- (b) If $n > 1$, then $F_n = F_{n-2} + F_{n-1}$.



Be ready for the
exam on the next
lecture

References

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Thank You for
Listening.