Relations

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Introduction

- The definition of a set explicitly disregards the order of the set elements, all that matters is who's in, not who's in first. However, sometimes the order is important.
- This leads to the notion of an ordered pair of two elements x and y, denoted (x, y).
- The crucial property is:

$$(x, y) = (u, v) \text{ if and only if } x = u \text{ and } y = v.$$

Binary Relation

Definition Given sets A and B, a binary relation between A and B is a subset of A × B.

- where *A* is the *domain* and *B* is the *codomain (or range)* of *R*. For $x \in A$ and $y \in B$, the notation *xRy* means $(x, y) \in R$.

Example

Let P be the set of people *{Bill, Sue, Pat}*, and let A be the set of animals *{dog, cat}*. The relation ***has*** :: $P \times A$ describes which person has which kind of animal, as the following relational expressions:

Bill has dog | *Sue has cat*

Sue has dog | *Pat has cat*

but the statement '*Bill has cat*' is false. Written out in full, the relation is

$$has = \{(Bill, dog), (Sue, dog), (Sue, cat), (Pat, cat)\}$$

Example

- Let $A = B = \{1, 2, 3, 4, 5, 6\}$ and

  R = $\{(a, b) : a$ divides $b\}$. Since A is a small finite set we can list the elements of the relation:

R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)}.

Laws

- $A \times B \neq B \times A$.
- $n(A \times B) = n(A) \times n(B)$.

Product Sets

- Give two sets A and B, their cartesian product $A \times B$ is the set of all ordered pairs (x, y), such that $x \in A$ and $y \in B$:

$$A \times B = \{(x, y) : x \in A, y \in B\}.$$

- Here is a useful special case:

$$A^2 = A \times A = \{(x, y) : x, y \in A\}.$$

Product Sets (Cont.)

- Example

Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a\}, (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

$$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

Inverse Relation

- The inverse of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,

$$R^{-1}=\{(b,a) \mid (a,b) \in R\}$$

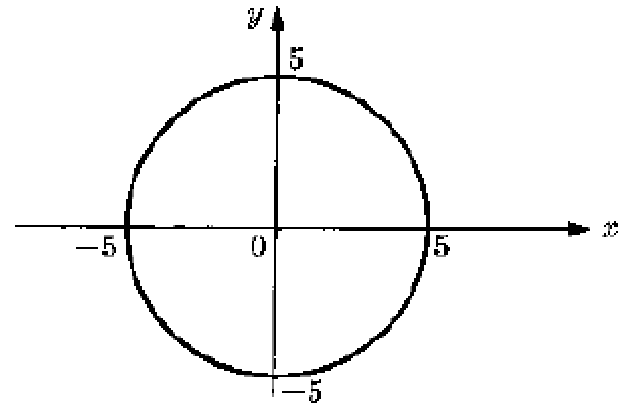
Example

$$Gt_{\mathbb{N}} = \{(1,0), (2,0), (2,1), (3,0), (3,1), (3,2), (4,0), \ldots\}$$
$$Gt_{\mathbb{N}}^{-1} = \{(0,1), (0,2), (1,2), (0,3), (1,3), (2,3), (0,4), \ldots\}$$

- Another example is on page 25 from the book.

Pictorial Representative of Relations

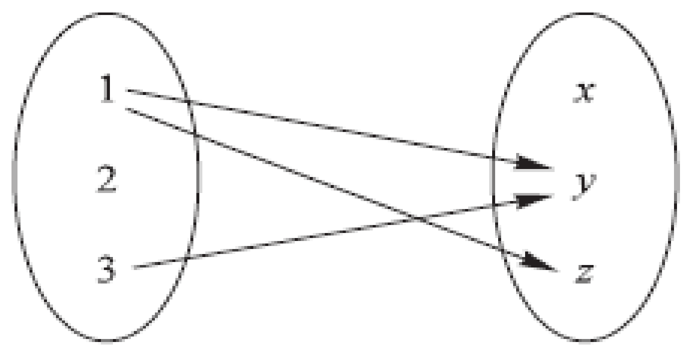
- The pictorial representation of the relation is sometimes called the graph of the relation.



$x^2 + y^2 = 25$

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

(i)



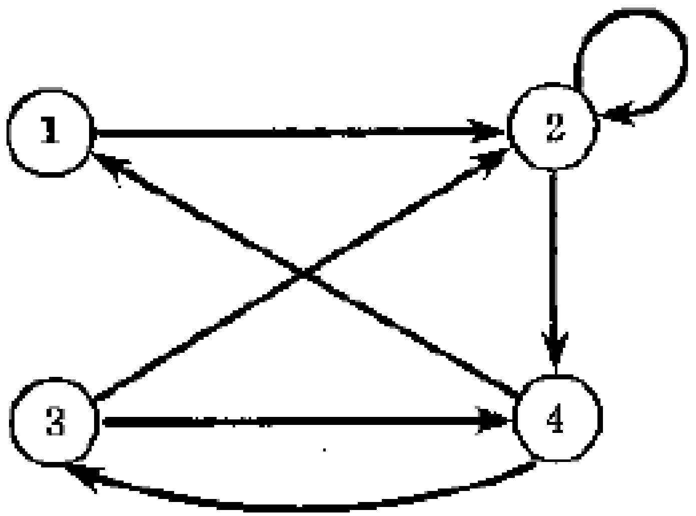
(ii)

$R = \{(1, y),\ (1, z),\ (3, y)\}$

Pictorial Representative of Relations (Cont.)

- Directed Graphs of Relations on Sets

$$R = \{(1,2),\ (2,2),\ (2,4),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\}$$



Composition of Relations

- This section is a homework.

Kinds of relations

- A relation R on a set A is called

• Reflexive if for all $a \in A$, aRa.
• Symmetric if for all $a, b \in A$, aRb implies bRa.
• Antisymmetric if for all $a, b \in A$, aRb and bRa implies $a = b$.
• Transitive if for all $a, b, c \in A$, aRb and bRc implies aRc.

Examples on kinds of relations

- Reflexive if for all a ∈ A, aRa.
- Which one is Reflexive if A={1,2,3,4}?

$$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$$

$$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$$

$$R_3 = \{(1,3), (2,1)\}$$

$$R_4 = \emptyset, \text{ the empty relation}$$

$$R_5 = A \times A, \text{ the universal relation}$$

- R_2 and R_5 are reflexive

Examples on kinds of relations (Cont.)

- Which one is Reflexive?

(1) Relation $\leq$ (less than or equal) on the set $\mathbf{Z}$ of integers
(2) Set inclusion $\subseteq$ on a collection $\mathcal{C}$ of sets
(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.
(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.

- R_1 and R_2 are reflexive.

Examples on kinds of relations (Cont.)

- **Example**

Let $R :: A \times A$ be a relation, where $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2)\}$. R is not reflexive, but if we added (3, 3) to R then it would be reflexive.

Examples on kinds of relations (Cont.)

- Symmetric if for all a, b ∈ A, aRb implies bRa.
- Which one is Symmetric?

$$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$$

$$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$$

$$R_3 = \{(1,3), (2,1)\}$$

$$R_4 = \emptyset, \text{ the empty relation}$$

$$R_5 = A \times A, \text{ the universal relation}$$

- R2, R4, and R5 are Symmetric.

Examples on kinds of relations (Cont.)

- Which one is Symmetric?

(1) Relation $\leq$ (less than or equal) on the set $\mathbf{Z}$ of integers
(2) Set inclusion $\subseteq$ on a collection $\mathcal{C}$ of sets
(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.
(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.

- R_3 and R_4 are Symmetric.

Examples on kinds of relations (Cont.)

- R is antisymmetric if $(y, x) \notin R$ whenever $(x, y) \in R$ and $x \neq y$.
- Which one is Antisymmetric?

$$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$$
$$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$$
$$R_3 = \{(1,3), (2,1)\}$$
$$R_4 = \emptyset, \text{ the empty relation}$$
$$R_5 = A \times A, \text{ the universal relation}$$

- R_1, R_3, and R_4 are Antisymmetric.

Examples on kinds of relations (Cont.)

- Transitive if for all a, b, c ∈ A, aRb and bRc implies aRc.
- Which one is transitive?

$$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$$
$$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$$
$$R_3 = \{(1,3), (2,1)\}$$
$$R_4 = \emptyset, \text{ the empty relation}$$
$$R_5 = A \times A, \text{ the universal relation}$$

- R_1, R_2, R_4, and R_5 are transitive.

Closure

- Let A be a set and R be a relation on set A. The reflexive / symmetric / transitive closure of R is a relation R* such that:

– R* is a reflexive / symmetric / transitive relation on A.

– $R \subseteq R^*$.

- Intuitively, a closure of R is the smallest extension of R to achieve a certain property (e.g., being reflexive / symmetric / transitive).

Closure example

- Suppose A={1,2,3}. Suppose R={(1,2), (2,3)} is a relation on A.

– The reflexive closure of R is

{(1,2),(2,3),(1,1),(2,2),(3,3)}.

– The symmetric closure of R is

{(1,2),(2,3),(2,1),(3,2)}.

– The transitive closure of R is

{(1,2), (2,3), (1,3)}.

Calculating Closure

- Given a set A and a relation R on A, how can we calculate the reflexive / symmetric / transitive closure of R?

– The reflexive closure of R can be calculated by adding (a,a) to R for every a ∈A (unless (a,a) is already in R).

– The symmetric closure of R can be calculated by adding (b,a) to R for every (a,b) ∈ R (unless (b,a) is already in R).

Let R be a relation on a set A. Then:

(i) $R \cup \Delta_A$ is the reflexive closure of R.

(ii) $R \cup R^{-1}$ is the symmetric closure of R.

Calculating Closure

– But calculating the transitive closure is more challenging. Refer to page 31 on the book for more details.

Equivalence Relations

- A relation R on a set *A* is an **equivalence relation** if R is reflexive, symmetric and transitive.
- More examples are in page 31 of the book.

Other Subsections…

- Equivalence Relations & Partitions: This subsection is now discarded.
- Partial Ordering Relations: This subsection will be considered later in more details.
- *n*-Ary relation: This subsection is out the scope of this chapter.

Be ready for the exam on the next lecture

References

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Thank You for Listening.