The Laplace transform

Example 3. $f(t) = t^n$, for $n \geq 1$ integer.

$$\begin{aligned} F(s) &= \lim_{A\to\infty} \int_0^A e^{-st} t^n dt = \lim_{A\to\infty} \left\{ t^n \frac{e^{-st}}{-s}\Big|_0^A - \int_0^A \frac{nt^{n-1}e^{-st}}{-s} dt \right\} \\ &= 0 + \frac{n}{s} \lim_{A\to\infty} \int_0^A e^{-st} t^{n-1} dt = \frac{n}{s} \mathcal{L}\{t^{n-1}\}. \end{aligned}$$

So we get a recursive relation

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}, \quad \forall n,$$

which means

$$\mathcal{L}\{t^{n-1}\} = \frac{n-1}{s}\mathcal{L}\{t^{n-2}\}, \quad \mathcal{L}\{t^{n-2}\} = \frac{n-2}{s}\mathcal{L}\{t^{n-3}\}, \cdots$$

By induction, we get

$$\begin{aligned} \mathcal{L}\{t^n\} &= \frac{n}{s}\mathcal{L}\{t^{n-1}\} = \frac{n}{s}\frac{(n-1)}{s}\mathcal{L}\{t^{n-2}\} = \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\mathcal{L}\{t^{n-3}\} \\ &= \cdots = \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\cdots\frac{1}{s}\mathcal{L}\{1\} = \frac{n!}{s^n}\frac{1}{s} = \frac{n!}{s^{n+1}}, \quad (s > 0) \end{aligned}$$

Example 4. Find the Laplace transform of $\sin at$ and $\cos at$.

Method 1. Compute by definition, with integration-by-parts, twice. (lots of work...)

Method 2. Use the Euler's formula

$$e^{iat} = \cos at + i \sin at, \qquad \Rightarrow \qquad \mathcal{L}\{e^{iat}\} = \mathcal{L}\{\cos at\} + i\mathcal{L}\{\sin at\}.$$

By Example 2 we have

$$\mathcal{L}\{e^{iat}\} = \frac{1}{s-ia} = \frac{1(s+ia)}{(s-ia)(s+ia)} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}.$$

Comparing the real and imaginary parts, we get

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \qquad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \quad (s > 0).$$

Remark: Now we will use $\int_0^\infty$ instead of $\lim_{A\to\infty}\int_0^A$, without causing confusion.

For piecewise continuous functions, Laplace transform can be computed by integrating each integral and add up at the end.

Example 5. Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ t-2, & 2 \leq t. \end{cases}$$