11.2 Functions of Any Period $p = 2L$

The functions considered so far had period 2π, for the simplicity of the formulas. Of course, periodic functions in applications will generally have other periods. However, we now show that the transition from period $p = 2\pi$ to a period $2L$ is quite simple. The notation $p = 2L$ is practical because L will be the length of a violin string (Sec. 12.2) or the length of a rod in heat conduction (Sec. 12.5), and so on.

The idea is simply to find and use a ***change of scale*** that gives from a function $g(v)$ of period 2π a function of period $2L$. Now from (5) and (6) in the last section with $g(v)$ instead of $f(x)$ we have the Fourier series

$$g(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv) \tag{1}$$

with coefficients

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v)\, dv \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \cos nv\, dv \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \sin nv\, dv. \end{aligned} \tag{2}$$

We can now write the change of scale as $v = kx$ with k such that the old period $v = 2\pi$ gives for the new variable x the new period $x = 2L$. Thus, $2\pi = k2L$. Hence $k = \pi/L$ and

$$v = kx = \pi x/L. \tag{3}$$

This implies $dv = (\pi/L)\, dx$, which upon substitution into (2) cancels $1/2\pi$ and $1/\pi$ and gives instead the factors $1/2L$ and $1/L$. Writing

$$g(v) = f(x), \tag{4}$$

we thus obtain from (1) the **Fourier series** of the function $f(x)$ of period $2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \tag{5}$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**

$$\begin{aligned} &\text{(a)} \quad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \\ &\text{(b)} \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots \\ &\text{(c)} \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots . \end{aligned} \tag{6}$$

Just as in Sec. 11.1, we continue to call (5) with any coefficients a **trigonometric series.** And we can integrate from 0 to $2L$ or over any other interval of length $p = 2L$.

EXAMPLE 1 Periodic Rectangular Wave

Find the Fourier series of the function (Fig. 259)

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } \quad 1 < x < 2 \end{cases} \qquad p = 2L = 4, \quad L = 2.$$

Solution. From (6a) we obtain $a_0 = k/2$ (verify!). From (6b) we obtain

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}.$$

Thus $a_n = 0$ if n is even and

$$a_n = 2k/n\pi \quad \text{if} \quad n = 1, 5, 9, \cdots, \qquad a_n = -2k/n\pi \quad \text{if} \quad n = 3, 7, 11, \cdots.$$

From (6c) we find that $b_n = 0$ for $n = 1, 2, \cdots$. Hence the Fourier series is

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right).$$

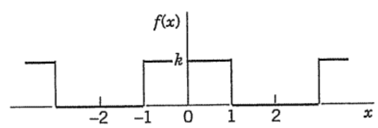

Fig. 259. Example 1

EXAMPLE 2 Periodic Rectangular Wave

Find the Fourier series of the function (Fig. 260)

$$f(x) = \begin{cases} -k & \text{if } -2 < x < 0 \\ k & \text{if } \quad 0 < x < 2 \end{cases} \qquad p = 2L = 4, \qquad L = 2.$$

Solution. $a_0 = 0$ from (6a). From (6b), with $1/L = 1/2$,

$$\begin{aligned} a_n &= \frac{1}{2}\left[\int_{-2}^{0} (-k)\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} k\cos\frac{n\pi x}{2}\,dx\right] \\ &= \frac{1}{2}\left[-\frac{2k}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{0} + \frac{2k}{n\pi}\sin\frac{n\pi x}{2}\Big|_{0}^{2}\right] = 0, \end{aligned}$$

so that the Fourier series has no cosine terms. From (6c),

$$\begin{aligned} b_n &= \frac{1}{2}\left[\frac{2k}{n\pi}\cos\frac{n\pi x}{2}\Big|_{-2}^{0} - \frac{2k}{n\pi}\cos\frac{n\pi x}{2}\Big|_{0}^{2}\right] \\ &= \frac{k}{n\pi}(1 - \cos n\pi - \cos n\pi + 1) = \begin{cases} 4k/n\pi & \text{if } n = 1, 3, \cdots \\ 0 & \text{if } n = 2, 4, \cdots. \end{cases} \end{aligned}$$