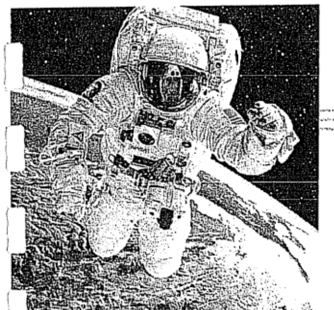

CHAPTER 11

Fourier Series, Integrals, and Transforms

Fourier series (Sec. 11.1) are infinite series designed to represent general periodic functions in terms of simple ones, namely, cosines and sines. They constitute a very important tool, in particular in solving problems that involve ODEs and PDEs.

In this chapter we discuss Fourier series and their engineering use from a practical point of view, in connection with ODEs and with the approximation of periodic functions. Application to PDEs follows in Chap. 12.

The *theory* of Fourier series is complicated, but we shall see that the *application* of these series is rather simple. Fourier series are in a certain sense more universal than the familiar Taylor series in calculus because many *discontinuous* periodic functions of practical interest can be developed in Fourier series but, of course, do not have Taylor series representations.

In the last sections (11.7–11.9) we consider **Fourier integrals** and **Fourier transforms,** which extend the ideas and techniques of Fourier series to nonperiodic functions and have basic applications to PDEs (to be shown in the next chapter).

Prerequisite: Elementary integral calculus (needed for Fourier coefficients)
Sections that may be omitted in a shorter course: 11.4–11.9
References and Answers to Problems: App. 1 Part C, App. 2.

11.1 Fourier Series

Fourier series are the basic tool for representing periodic functions, which play an important role in applications. A function $f(x)$ is called a **periodic function** if $f(x)$ is defined for all real x (perhaps except at some points, such as $x = \pm\pi/2, \pm 3\pi/2, \cdots$ for $\tan x$) and if there is some positive number p, called a **period** of $f(x)$, such that

$$f(x + p) = f(x) \qquad \text{for all } x. \tag{1}$$

The graph of such a function is obtained by periodic repetition of its graph in any interval of length p (Fig. 255).

Familiar periodic functions are the cosine and sine functions. Examples of functions that are not periodic are $x, x^2, x^3, e^x, \cosh x$, and $\ln x$, to mention just a few.

If $f(x)$ has period p, it also has the period $2p$ because (1) implies $f(x + 2p) = f([x + p] + p) = f(x + p) = f(x)$, etc.; thus for any integer $n = 1, 2, 3, \cdots$,

$$f(x + np) = f(x) \qquad \text{for all } x. \tag{2}$$

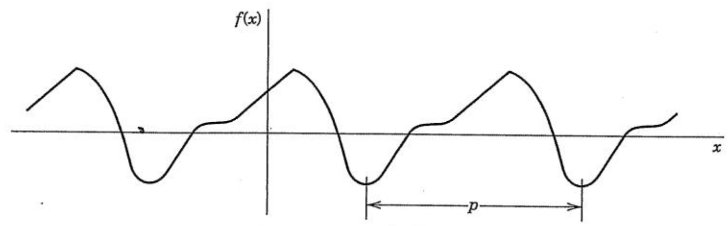

Fig. 255. Periodic function

Furthermore if $f(x)$ and $g(x)$ have period p, then $af(x) + bg(x)$ with any constants a and b also has the period p.

Our problem in the first few sections of this chapter will be the representation of various ***functions $f(x)$ of period 2π*** in terms of the simple functions

(3) $1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \cdots, \quad \cos nx, \quad \sin nx, \cdots.$

All these functions have the period 2π. They form the so-called **trigonometric system.** Figure 256 shows the first few of them (except for the constant 1, which is periodic with any period).

The series to be obtained will be a **trigonometric series,** that is, a series of the form

$$(4) \qquad \begin{aligned} &a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots \\ &= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \end{aligned}$$

$a_0, a_1, b_1, a_2, b_2, \cdots$ are constants, called the **coefficients** of the series. We see that each term has the period 2π. Hence *if the coefficients are such that the series converges, its sum will be a function of period 2π.*

It can be shown that if the series on the left side of (4) converges, then inserting parentheses on the right gives a series that converges and has the same sum as the series on the left. This justifies the equality in (4).

Now suppose that $f(x)$ is a given function of period 2π and is such that it can be **represented** by a series (4), that is, (4) converges and, moreover, has the sum $f(x)$. Then, using the equality sign, we write

$$(5) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

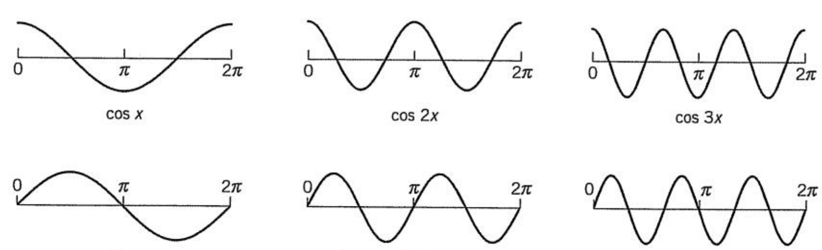

Fig. 256. Cosine and sine functions having the period 2π