Matrices

A rectangular array of numbers of the form $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ is called $m \times n$ matrix, with m rows and n columns. We count rows from the top and columns from the left.

Matrices addition: Suppose that the two matrices

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

both have m rows and n columns. Then we write

$$A + B = \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

and call this the sum of the two matrices A and B.

Example 1: Suppose that $A = \begin{pmatrix} 1 & -2 \\ 3 & 0 \\ -5 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -1 & 6 \\ 5 & -4 \end{pmatrix}$

Then $$A + B = \begin{pmatrix} 1+2 & -2+3 \\ 3+(-1) & 0+6 \\ -5+5 & 7+(-4) \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 6 \\ 0 & 3 \end{pmatrix} \square$$

Remark: Two matrices of different sizes cannot be added.

Example 2: We do not have a definition for adding the matrices

$$A = \begin{pmatrix} 4 & 2 & -1 \\ 0 & 1 & 3 \\ 6 & -2 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 5 \end{pmatrix} \square$$

Multiplication by scalar : Suppose that the matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ has m rows and n columns, and that $c \in R$. Then we write

$$c\,A = \begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \vdots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{pmatrix}$$

Example 3: Suppose that

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 1 & 2 & 4 \end{pmatrix} \quad \text{then} \quad 3A = \begin{pmatrix} 9 & 0 & -3 \\ 3 & 6 & 12 \end{pmatrix}$$

Matrix multiplication: Suppose that

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \vdots & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix}$$

are respectively $m \times n$ matrix and $n \times p$ matrix. Then the product AB will make sense and it will be $m \times p$ matrix.

$$AB=\begin{pmatrix} q_{11} & \cdots & q_{1p} \\ \vdots & \vdots & \vdots \\ q_{m1} & \cdots & q_{mp} \end{pmatrix},$$

where for every $i=1,2,\cdots,m$ and $j=1,2,\cdots,p,$ we defined

$$q_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$$

Example 4: Find AB and BA if □

$$A=\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix} 2 & 1 \\ 4 & -3 \\ -1 & 0 \end{pmatrix}$$

Solution: AB is going to make sense. It is the product of 2×3 by 3×2 and the result is going to be 2×2.

$$AB=\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

$q_{11}=1\times 2+(-1)\times 4+2\times(-1)=-4$

$q_{12}=1\times 1+(-1)\times(-3)+2\times 0=4$

$q_{21}=3\times 2+0\times 4+(-2)\times(-1)=8$

$q_{22}=3\times 1+0\times(-3)+(-2)\times 0=3$

$$AB=\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & -3 \\ -1 & 0 \end{pmatrix}=\begin{pmatrix} -4 & 4 \\ 8 & 3 \end{pmatrix}$$

$$BA=\begin{pmatrix} 2 & 1 \\ 4 & -3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix}$$

$$=\begin{pmatrix} 2\times 1+1\times 3 & 2\times(-1)+1\times 0 & 2\times 2+1\times(-2) \\ 4\times 1+(-3)\times 3 & 4\times(-1)+(-3)\times 0 & 4\times 2+(-3)\times(-2) \\ (-1)\times 1+0\times 3 & (-1)\times(-1)+0\times 0 & (-1)\times 2+0\times(-2) \end{pmatrix}$$

$$=\begin{pmatrix} 5 & -2 & 2 \\ -5 & -4 & 12 \\ -1 & 1 & -2 \end{pmatrix}$$

Transpse of a Matrix:

Let A be $m \times n$ matrix. Then A^T,the transpse of A is the matrix obtained by interchanging the rows and columns of A.

For example:

$$\text{If } A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 5 \end{pmatrix} \quad \text{then } A^T = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ -1 & 5 \end{pmatrix}$$

Determinants:

If A is $n \times n$ square matrix is called a matrix of order n.With such a matrix we associate a number called the determinant of A and written $det\, A$.

For $n = 2$ we have this definition:

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ then } det\, A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{21} \times a_{12}$$

For $n = 3$ we have this definition:(we will choose first row)

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ then } det\, A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{array}$$

$$det\, A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Remark : we can choose any row or any column of A.

Example 6: Find $det\, A$ for

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{pmatrix}$$

Solution:

$$det\, A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$det\, A = 2 \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} - 1 \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$$

$$det\, A = 2\{(-1) \times 1 - 3 \times (-2)\} - 1\{3 \times 1 - 2 \times (-2)\} + 3\{3 \times 3 - 2 \times (-1)\}$$

$$det\, A = 2 \times 5 - 7 + 3 \times 11 = 36$$

Minors and Cofactors

Suppose we are given a matrix A of order n , and a_{ij} is one of its entries. The minor M_{ij} is defined to be the determinant of the matrix obtained by deleting the row i and the column j from the matrix.

The cofactor $A_{ij} = (-1)^{i+j} M_{ij}$

Example 7: Find the cofactor of a_{23} if $A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{pmatrix}$

$$A_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = -(6-2) = -4$$

Adjoint of Matrix

The adjoint of a matrix A of order n is obtained by taking the transpose of the cofactor matrix A and we write $adj(A)$.

To find the adjoint of a matrix, first find the cofactor matrix of the given matrix. Then find the transpose of the cofactor matrix.

Example 8: Find $adj(A)$ for $A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{pmatrix}$

Solution:

$A_{11} = (-1)^{1+1} \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} = 5$ $\qquad$ $A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = -7$

$A_{13} = (-1)^{1+3} \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} = 11$ $\qquad$ $A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -4$ $\qquad$ $A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = -4$

$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix} = 1$ $\qquad$ $A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = 13$

$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -5$

$$A_{ij} = \begin{pmatrix} 5 & -7 & 11 \\ 8 & -4 & -4 \\ 1 & 13 & -5 \end{pmatrix}$$

$$adj(A) = A_{ij}^T = \begin{pmatrix} 5 & 8 & 1 \\ -7 & -4 & 13 \\ 11 & -4 & -5 \end{pmatrix}$$

Identity matrix

The identity matrix of order n denoted I_n is a diagonal matrix with all ones on the diagonal

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If A is an $m\ x\ n$ matrix, then

$$I_m A = A \quad \text{and} \quad A = AI_n$$

If A is a square matrix of order n , then

$$I_n A = AI_n = A$$

Inversion of matrices

If A is a square matrix of order n , then its inverse is given by

$$A^{-1} = \frac{1}{det(A)} adj(A) \quad ; \quad det(A) \neq 0.$$

The inverse has the special property that

$$AA^{-1} = A^{-1}A = I$$

Example 9: Find A^{-1} for $A = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{pmatrix}$

Solution:

$$det(A) = 5(42-12) - 2(0-24) + 1(0-48) = 150$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 6 & 3 \\ 4 & 7 \end{vmatrix} = 30 \quad , \quad A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 3 \\ 8 & 7 \end{vmatrix} = 24$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 6 \\ 8 & 4 \end{vmatrix} = -48 \quad , \quad A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix} = -10$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 5 & 1 \\ 8 & 7 \end{vmatrix} = 27 \quad , \quad A_{23} = (-1)^{2+3} \begin{vmatrix} 5 & 2 \\ 8 & 4 \end{vmatrix} = -4$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = 0 \quad , \quad A_{32} = (-1)^{3+2} \begin{vmatrix} 5 & 1 \\ 0 & 3 \end{vmatrix} = -15$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 5 & 2 \\ 0 & 6 \end{vmatrix} = 30$$

$$A_{ij} = \begin{pmatrix} 30 & 24 & -48 \\ -10 & 27 & -4 \\ 0 & -15 & 30 \end{pmatrix}$$

$$adj(A) = \begin{pmatrix} 30 & -10 & 0 \\ 24 & 27 & -15 \\ -48 & -4 & 30 \end{pmatrix}□$$

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

$$A^{-1} = \frac{1}{150} \begin{pmatrix} 30 & -10 & 0 \\ 24 & 27 & -15 \\ -48 & -4 & 30 \end{pmatrix}□$$

Exercises

If □

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 3 \\ 0 & 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -1 \\ 2 & 1 & -2 \end{pmatrix}$$

1. Find AB and BA
2. Find A^{-1} and B^{-1}