Eigenvalues and Eigenvectors

Let A be $n \times n$ matrix. A scalar λ is said to be an eigenvalue of A ,if $AX = \lambda X$ for some vector $X \neq 0$.

The scalar λ is an eigenvalue of A if and only if $det(\lambda I - A) = 0$.

And the vector X is an eigenvector , of A, corresponding to λ if and only if X is a nonzero solution $(\lambda I - A)X = 0$.

Example 1: Find eigenvalue and eigenvector of $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

Solution:

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} \lambda - 4 & -1 \\ -3 & \lambda - 2 \end{pmatrix}$$

$$det(\lambda I - A) = 0 \quad \Rightarrow \quad \begin{vmatrix} \lambda - 4 & -1 \\ -3 & \lambda - 2 \end{vmatrix} = 0$$

$$(\lambda - 4)(\lambda - 2) - 3 = 0 \quad \Rightarrow \quad \lambda^2 - 6\lambda + 8 - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0 \quad \Rightarrow \quad (\lambda - 5)(\lambda - 1) = 0$$

So the eigenvalues are $\lambda = 5$ *and* $\lambda = 1$.

To find an eigenvector corresponding to $\lambda = 5$ we have to solve

$(5I - A)X = 0$ or

$$\begin{pmatrix} 5-4 & -1 \\ -3 & 5-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x - y = 0 \quad \Rightarrow \quad x = y \quad (\text{put} \quad x = t)$$

So the eigenvector is $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

To find an eigenvector corresponding to $\lambda = 1$ we have to solve

$(I - A)X = 0$ or

$$\begin{pmatrix} 1-4 & -1 \\ -3 & 1-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} -3 & -1 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-3x - y = 0 \quad \Rightarrow \quad -3x = y \quad (\text{put} \quad x = t)$$

So the eigenvector is $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -3t \end{pmatrix} = t \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Example 2: Find eigenvalue and eigenvector of $A = \begin{pmatrix} -5 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & -2 & 3 \end{pmatrix}$.

Solution:

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -5 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & -2 & 3 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} -5 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda + 5 & 0 & 0 \\ -3 & \lambda - 7 & 0 \\ -4 & 2 & \lambda - 3 \end{pmatrix}$$

$$det(\lambda I - A) = 0 \quad \Rightarrow \quad \begin{vmatrix} \lambda + 5 & 0 & 0 \\ -3 & \lambda - 7 & 0 \\ -4 & 2 & \lambda - 3 \end{vmatrix} = 0$$

$$(\lambda + 5)(\lambda - 7)(\lambda - 3) = 0$$

So the eigenvalues are $\lambda = -5$, $\lambda = 7$ and $\lambda = 3$.

To find an eigenvector corresponding to $\lambda = -5$ we have to solve

$$(-5I - A)X = 0 \quad \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ -3 & -12 & 0 \\ -4 & 2 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x - 12y = 0 \quad \Rightarrow \quad -3x = 12y \quad \Rightarrow x = -4y$$

and $\quad -4x + 2y - 8z = 0 \quad \Rightarrow \quad 18y = 8z \quad \Rightarrow z = (9/4)y$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4y \\ y \\ (9/4)y \end{pmatrix} \quad (\text{put} \quad y = t) \quad \Rightarrow \quad X = t \begin{pmatrix} -4 \\ 1 \\ 9/4 \end{pmatrix}$$

When $\quad \lambda = 7 \quad \Rightarrow \quad (7I - A)X = 0 \Rightarrow \begin{pmatrix} 12 & 0 & 0 \\ -3 & 0 & 0 \\ -4 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$x = 0$, $\quad -4x + 2y - 4z = 0 \quad \Rightarrow \quad y = 2z \quad (\text{put} \quad z = t)$

$$X = t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

Exercises

Find eigenvalue and eigenvector of $A = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}$