II-Power series method

We consider the second-order linear homogeneous differential equation for $y = y(x)$

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials. The idea of power series method is to assume that the unknown function y can be expanded into a power series:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + \cdots$$

Substituting in the differential equation and requiring that the coefficients of each power of x sum to zero, we can find all the constants $a_n \; \forall n = 2,3,4,\cdots$ in terms of a_0 or a_1.

Example 1: Solve $y'' + 2xy' + 2y = 0$ by power series method

Solution :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + \cdots$$

$$2xy' = 0 \quad + 2a_1 x + 4a_2 x^2 + 6a_3 x^3 \quad + 8a_4 x^4 \quad + 12a_5 x^5 + \cdots$$

$$2y = 2a_0 \quad + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 \quad + 2a_4 x^4 \quad + 2a_5 x^5 + \cdots$$

$$2a_2 + 2a_0 = 0 \quad \Rightarrow \boxed{a_2 = -a_0}$$

$$6a_3 + 2a_1 + 2a_1 = 0 \Rightarrow \boxed{a_3 = -(2/3)a_1}$$

$$12a_4 + 4a_2 + 2a_2 = 0 \quad \Rightarrow \quad a_4 = -(1/2)a_2 \quad \Rightarrow \boxed{a_4 = (1/2)a_0}$$

$$20a_5 + 6a_3 + 2a_3 = 0 \quad \Rightarrow \quad a_5 = -(2/5)a_3 \quad \Rightarrow \quad a_5 = (4/15)a_1$$

$$y = a_0 + a_1 x - a_0 x^2 - (2/3)a_1 x^3 + (1/2)a_0 x^4 + (4/15)a_1 x^5 + \cdots$$

$$y = a_0\left(1 - x^2 + \frac{1}{2}x^4 + \cdots\right) + a_1\left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \cdots\right)$$

Example 2: Solve $y'' - xy' + 4y = 0$ by using power series method

Solution :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + \cdots$$

Now

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + \cdots$$

$$-xy' = 0 - a_1 x - 2a_2 x^2 - 3a_3 x^3 - 4a_4 x^4 - 5a_5 x^5 - \cdots$$

$$4y = 4a_0 + 4a_1 x + 4a_2 x^2 + 4a_3 x^3 + 4a_4 x^4 + 4a_5 x^5 + \cdots$$

$$2a_2 + 4a_0 = 0 \Rightarrow \boxed{a_2 = -2a_0} \quad , \quad 6a_3 - a_1 + 4a_1 = 0 \Rightarrow \boxed{a_3 = -(1/2)a_1}$$

$$12a_4 - 2a_2 + 4a_2 = 0 \Rightarrow a_4 = -(1/6)a_2 \Rightarrow \boxed{a_4 = (1/3)a_0}$$

$$20a_5 - 3a_3 + 4a_3 = 0 \Rightarrow a_5 = -(1/20)a_3 \Rightarrow \boxed{a_5 = (1/40)a_1}$$

$$30a_6 - 4a_4 + 4a_4 = 0 \Rightarrow \boxed{a_6 = 0} \quad \text{and} \quad a_8 = a_{10} = \cdots = 0$$

$$y = a_0 + a_1 x - 2a_0 x^2 - (1/2)a_1 x^3 + (1/3)a_0 x^4 + (1/40)a_1 x^5 + \cdots$$

$$y = a_0\left(1 - 2x^2 + \frac{x^4}{3}\right) + a_1\left(x - \frac{x^3}{2} + \frac{x^5}{40} + \cdots\right)$$

Exercises

Solve the differential equations by using power series method

(1) $x y'' + y' + xy = 0$

(2) $x^2 y'' + y' + x^2 y = 0$

(3) $y'' + 3xy' - y = 0$

(4) $(x-1)y'' + 2y' = 0$