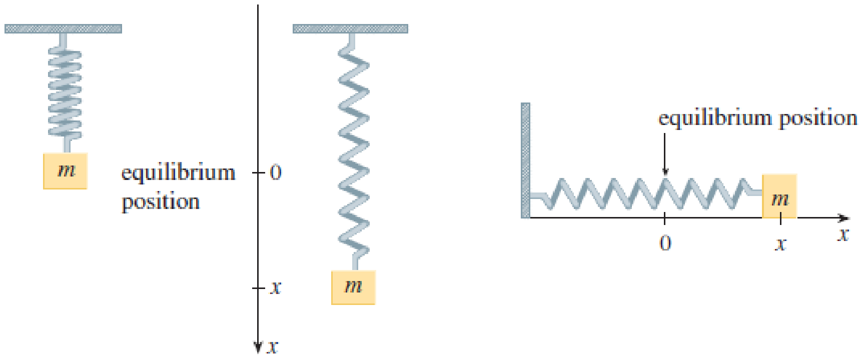
Physical Applications of Second-Order Differential Equations

Simple Harmonic Motion

We consider the motion of an object with mass at the end of a spring that is either vertical or horizontal on a level surface as in a figure .



Hooke's Law, says that if the spring is stretched units from its natural length, then it exerts a force that is proportional to x : restoring force $= -kx$

where k is a positive constant(called the spring constant). If we ignore any external resisting forces then, by Newton's Second Law we have

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad m\frac{d^2x}{dt^2} + kx = 0$$

Example 1: A frictionless spring with a 10kg mass can be held stretched 1 meters beyond its natural length by a force of 40 N. If the spring begins at its equilibrium position, but a push gives it an initial velocity of 2.5 m/sec, find the position of the mass after t seconds.

Solution: From Hooke's Law, the force required to stretch the spring is

$$40 = 1k \Rightarrow k = 40$$

$$10x'' + 40x = 0 \ ; \ x(0) = 1 \text{ and } x'(0) = 2.5$$

$$10r^2 + 40 = 0 \Rightarrow r = \mp 2i$$

$$x = c_1 \sin 2t + c_2 \cos 2t$$

$$x(0) = 1 \Rightarrow c_2 = 1$$

$$x' = 2c_1 \cos 2t - 2c_2 \sin 2t$$

$$x'(0) = 2.5 \Rightarrow 2.5 = 2c_1 \Rightarrow c_1 = 1.25$$

$$x = 1.25 \sin 2t + \cos 2t$$

Damped Vibrations

We next consider the motion of a spring that is subject to a frictional force .An example is the damping force supplied by a shock absorber in a car or a bicycle. We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. Thus:

$$\text{damping force} = -c\frac{dx}{dt}$$

where c is a positive constant, called the damping constant. Thus, in this case, Newton's Second Law gives :

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Example 2: A spring with a mass of $2\ kg$ has natural length $0.5\ m$. A force of $25.6\ N$ is required to maintain it stretched to a length of $0.7\ m$. If the spring is immersed in a fluid with damping constant $c = 40$. Find the position of the mass at any time if it starts from the equilibrium position and is given a push to start it with an initial velocity of $0.6\ m/s$.

Solution: $25.6 = 0.2k \;\Rightarrow\; k = 128,\ m = 2$ and $c = 40$

$$2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x = 0$$

$$2x'' + 40x' + 128x = 0 \;\; ; \;\; x(0) = 0 \text{ and } x'(0) = 0.6$$

$$x'' + 20x' + 64x = 0$$

$$r^2 + 20r + 64 = 0 \;\Rightarrow (r+4)(r+16) = 0 \;\Rightarrow r = -4, -16$$

$$x(t) = c_1e^{-4t} + c_2e^{-16t}$$

$$x(0) = 0 \;\Rightarrow c_1 + c_2 = 0 \;\cdots(1)$$

$$x'(t) = -4c_1e^{-4t} - 16c_2e^{-16t}$$

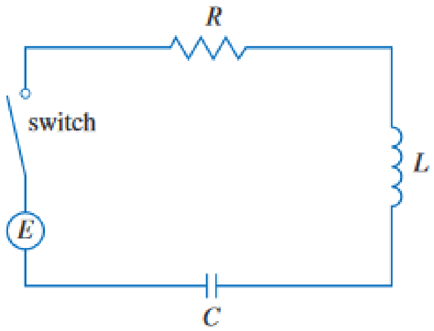
$$-4c_1 - 16c_2 = 0.6 \qquad \cdots(2)$$

$$c_1 = 0.05 \text{ and } c_2 = -0.05$$

$$x(t) = 0.05\ (e^{-4t} - e^{-16t})$$

Electric Circuits

The circuit shown in Figure contains an electromotive force E, a resistor R , an inductor L, and a capacitor C, in series. If the charge on the capacitor at time t is $Q = Q(t)$, then the current is the rate of change of Q with respect to t : $I = dQ/dt$.



It is known from physics that the voltage drops across the resistor, inductor, and capacitor are

$$RI \qquad L\frac{dI}{dt} \qquad \frac{Q}{C}$$

respectively. Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$$

Since $I = dQ/dt$, this equation becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t)$$

Example 3: A series circuit consists of a resistor with $R = 20\,\Omega$, an inductor with $L = 1\,H$, a capacitor with $C = 0.002\,F$, and a 12-V battery. If the initial charge and current are both 0, find the charge at time t.

. Solution:

$$\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + \frac{Q}{0.002} = 12$$

$$\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + 500\ Q = 12$$

$$r^2 + 20r + 500 = 0$$

$$r = \frac{-20 \mp \sqrt{400 - 2000}}{2} = -10 \mp 20i$$

$Q_h = e^{-10t}(c_1 \sin 20t + c_2 \cos 20t)$

Now $Q_p = A \Rightarrow \frac{dQ}{dt} = \frac{d^2Q}{dt^2} = 0$

So $500\, Q_p = 12 \quad \Rightarrow \quad Q_p = \frac{3}{125}$

$Q(t) = e^{-10t}(c_1 \sin 20t + c_2 \cos 20t) + \frac{3}{125}$

$Q(0) = 0 \quad \Rightarrow \quad 0 = c_2 + \frac{3}{125} \quad \Rightarrow \quad c_2 = -\frac{3}{125}$

$I = \frac{dQ}{dt} = e^{-10t}(20c_1 \cos 20t - 20c_2 \sin 20t) - 10e^{-10t}(c_1 \sin 20t + c_2 \cos 20t)$

$I(0) = 0 \quad \Rightarrow \quad 0 = 20c_1 - 10c_2 \quad \Rightarrow c_1 = -\frac{3}{250}$

$Q(t) = e^{-10t}\left(-\frac{3}{250} \sin 20t - \frac{3}{125} \cos 20t\right) + \frac{3}{125}$

Exercises

1. A spring with a mass of 3 kg has damping constant $c = 30$ and spring constant $k = 123$. Find the position of the mass at time if it starts at the equilibrium position with a velocity of $2\ m/s$.

2. A series circuit consists of a resistor with $R = 24\ \Omega$, an inductor with $L = 2\ H$, a capacitor with $C = 0.005\ F$, and a 12-V battery. If $Q(0) = 0.001$ and $I(0) = 0$, find the charge at time t.