Vector Functions

A vector function is a function that takes one or more variables and returns a vector. We say that $F(x,y,z)$ is a vector function or vector field if

$$F(x,y,z) = F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k.$$

For example $F(x,y,z) = x^2yz\, i + 2xy\, j + xy\sec z\;\; k$

1. Gradient of a scalar field

A scalar field is a function that takes a point in space and assign a number to it , for example $f(x,y,z) = x^2 + \cos 2y + \ln(2z+1)$

$$f\left(1,\frac{\pi}{6},0\right) = 1 + \cos\frac{\pi}{3} + \ln 1 = \frac{3}{2}$$

The gradient of a given scalar field $f(x,y,z)$ is a vector field denoted by $grad\ f$ or ∇f and it is defined as follows :

$$\nabla f = \frac{\partial f}{\partial x}\, i + \frac{\partial f}{\partial y}\, j + \frac{\partial f}{\partial z}\, k$$

Example 1: Find ∇f for $f = 2x^2\sin y - xy\tan z$

Solution :

$$\frac{\partial f}{\partial x} = 4x\sin y - y\tan z\square$$

$$\frac{\partial f}{\partial y} = 2x^2\cos y - x\tan z\square$$

$$\frac{\partial f}{\partial z} = -xy\sec^2 z \qquad \square$$

$$\nabla f = \frac{\partial f}{\partial x}\, i + \frac{\partial f}{\partial y}\, j + \frac{\partial f}{\partial z}\, k\square$$

$$\nabla f = (4x\sin y - y\tan z)\, i + (2x^2\cos y - x\tan z)\, j - xy\sec^2 z\;\; k\square$$

2. Laplace operator

The differential operator ∇^2 is called *Laplace operator* and it is defined as follows

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Example 2: Find ∇f and $\nabla^2 f$ for $f = x^3 e^y + xy^2 z^3$ at (3,0,2)

Solution :

$$\frac{\partial f}{\partial x} = 3x^2 e^y + y^2 z^3 \quad \Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = 6xe^y \quad \square$$

$$\frac{\partial f}{\partial y} = x^3 e^y + 2xyz^3 \quad \Rightarrow \quad \frac{\partial^2 f}{\partial y^2} = x^3 e^y + 2xz^3 \square$$

$$\frac{\partial f}{\partial z} = 3xy^2 z^2 \quad \Rightarrow \quad \frac{\partial^2 f}{\partial z^2} = 6xy^2 z \quad \square$$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = (3x^2 e^y + y^2 z^3)\, i + (x^3 e^y + 2xyz^3)\, j + 3xy^2 z^2\, k \square$$

$$\nabla f|_{\text{at }(3,0,2)} = (3(3)^2 e^0 + (0)^2 (2)^3)\, i + ((3)^3 e^0 + 2(3)(0)(2)^3)\, j + 3(3)(0)^2 (2)^2\, k \square$$

$$\nabla f|_{\text{at }(3,0,2)} = 27i + 27j \square$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 6xe^y + x^3 e^y + 2xz^3 + 6xy^2 z$$

$$\nabla^2 f|_{\text{at }(3,0,2)} = 18 + 27 = 81 \quad \square$$

3. Divergence of a vector field

The divergence of a vector field $F(x,y,z) = F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k$ is a scalar field $div\ F = \nabla.F$ and it is defined as follows

$$div\ F = \nabla.F == \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Example 3: Find $div\ F$ if $F(x,y,z) = xzi + e^{yz} j - \ln(xy)\, k$

Solution :

$$div\ F = \nabla.F = \frac{\partial (xz)}{\partial x} + \frac{\partial (e^{yz})}{\partial y} - \frac{\partial (\ln(xy))}{\partial z} = z + ze^{yz} \square$$

4. The curl of a vector field

The *curl of a vector field* $F(x,y,z) = F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k$ is another vector defined as the following determinant

$$curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)k \square$$

Example 4: Find $curl\ F$ if $F(x,y,z) = xz\sec y\, i + y\sin 2z\, j + y\cos 3x\, k$

Solution :

$$curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)k \square$$

$$= \left(\frac{\partial(y\cos 3x)}{\partial y} - \frac{\partial(y\sin 2z)}{\partial z}\right)i - \left(\frac{\partial(y\cos 3x)}{\partial x} - \frac{\partial(xz\sec y)}{\partial z}\right)j$$

$$+ \left(\frac{\partial(y\sin 2z)}{\partial x} - \frac{\partial(xz\sec y)}{\partial y}\right)k \square$$

$$= (\cos 3x - 2y\cos 2z)i - (-3y\sin 3x - x\sec y)j + (0 - xz\sec y\tan y)k \square$$

$$= (\cos 3x - 2y\cos 2z)i + (3y\sin 3x + x\sec y)j - (xz\sec y\tan y)k \qquad \square$$

Exercises

If $f(x,y,z) = x^3y^2z$ and $F(x,y,z) = yze^{xy}i + xze^{xy}j + (e^{xy} + 3\cos 3z)k.$

Then find

(1) ∇f at $(-1,2,-2)\square$

(2) $\nabla^2 f$ at $(1,-3,2)\square$

(3) $div\ F$ at $(0,\sqrt{6},\pi/6)\ \square$

(4) $curl\ F\ \square$