

Solving a system of linear equations by Cramer's rule

A system of three linear equations in three variables can be solved by using determinants in a manner similar to that of the previous section. This rule is also called Cramer's rule. The solution to the system

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Is given by

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D} \quad \text{and} \quad z = \frac{D_z}{D} \square$$

$$\text{Where } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},$$

$$\text{and } D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}, \quad \text{provided that } D \neq 0$$

Note that D_x , D_y and D_z are obtained from D by replacing the x -, y - or z -column with the constants d_1, d_2 and d_3 .

Example 1: Use Cramer's rule to solve the system

$$x + 2y - 3z = -2, \quad x - y + z = -1 \quad \text{and} \quad 3x + 4y - 4z = 4 \square$$

$$D = \begin{vmatrix} 1 & 2 & -3 \\ 1 & -1 & 1 \\ 3 & 4 & -4 \end{vmatrix} = 1 \begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix} + (-3) \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = -7 \square$$

$$D_x = \begin{vmatrix} -2 & 2 & -3 \\ -1 & -1 & 1 \\ 4 & 4 & -4 \end{vmatrix} = (-2) \begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} - 2 \begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} + (-3) \begin{vmatrix} -1 & -1 \\ 4 & 4 \end{vmatrix} = 0 \square$$

$$D_y = \begin{vmatrix} 1 & -2 & -3 \\ 1 & -1 & 1 \\ 3 & 4 & -4 \end{vmatrix} = 1 \begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix} + (-3) \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = -35 \square$$

$$D_z = \begin{vmatrix} 1 & 2 & -2 \\ 1 & -1 & -1 \\ 3 & 4 & 4 \end{vmatrix} = 1 \begin{vmatrix} -1 & -1 \\ 4 & 4 \end{vmatrix} - 2 \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} + (-2) \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = -28 \square$$

$$x = \frac{D_x}{D} = \frac{0}{-7} = 0, \quad y = \frac{D_y}{D} = \frac{-35}{-7} = 5 \quad \text{and} \quad z = \frac{D_z}{D} = \frac{-28}{-7} = 4$$

Example 2: Use Cramer's rule to solve the system

$$2x + 3y - z = 1, \quad 4x + y - 3z = 11 \text{ and } 3x - 2y + 5z = 21 \quad \square$$

$$D = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 1 & -3 \\ 3 & -2 & 5 \end{vmatrix} = 2 \begin{vmatrix} 1 & -3 \\ -2 & 5 \end{vmatrix} - 3 \begin{vmatrix} 4 & -3 \\ 3 & 5 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 1 \\ 3 & -2 \end{vmatrix} = -78 \quad \square$$

$$D_x = \begin{vmatrix} 1 & 3 & -1 \\ 11 & 1 & -3 \\ 21 & -2 & 5 \end{vmatrix} = 1 \begin{vmatrix} 1 & -3 \\ -2 & 5 \end{vmatrix} - 3 \begin{vmatrix} 11 & -3 \\ 21 & 5 \end{vmatrix} + (-1) \begin{vmatrix} 11 & 1 \\ 21 & -2 \end{vmatrix} = -312 \quad \square$$

$$D_y = \begin{vmatrix} 2 & 1 & -1 \\ 4 & 11 & -3 \\ 3 & 21 & 5 \end{vmatrix} = 2 \begin{vmatrix} 11 & -3 \\ 21 & 5 \end{vmatrix} - 1 \begin{vmatrix} 4 & -3 \\ 3 & 5 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 11 \\ 3 & 21 \end{vmatrix} = 156 \quad \square$$

$$D_z = \begin{vmatrix} 2 & 3 & 1 \\ 4 & 1 & 11 \\ 3 & -2 & 21 \end{vmatrix} = 2 \begin{vmatrix} 1 & 11 \\ -2 & 21 \end{vmatrix} - 3 \begin{vmatrix} 4 & 11 \\ 3 & 21 \end{vmatrix} + 1 \begin{vmatrix} 4 & 1 \\ 3 & -2 \end{vmatrix} = -78 \quad \square$$

$$x = \frac{D_x}{D} = \frac{-312}{-78} = 4, \quad y = \frac{D_y}{D} = \frac{156}{-78} = -2 \quad \text{and} \quad z = \frac{D_z}{D} = \frac{-78}{-78} = 1$$

Exercises

Use Cramer's rule to solve the system

$$(1) \quad x + 2y + 3z = 17$$

$$3x + 2y + z = 11$$

$$x - 5y + z = -5 \quad \square$$

$$(2) \quad x + 2y - 3z = -2$$

$$x - y + z = -1$$

$$3x + 4y - 4z = 4$$