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Distribution of the Function of Random Variable

Methods for Finding Distributions

Consider the random variables $Y_1, Y_2, \ldots, Y_n$ and a function $U(Y_1, Y_2, \ldots, Y_n)$, denoted simply as U. Then three of the methods for finding the probability distribution of U are as follows:

1. **The method of distribution functions:** This method is typically used when the $Y's$ have continuous distributions. First, find the distribution function for U, $F_U(u) = P(U \leq u)$, by using the methods that we discussed in Chapter 5. The density function for U is then obtained by differentiating the distribution function, $F_U(u)$.
2. **The method of transformations:** If we are given the density function for a random variable Y, the method of transformations results in a general expression for the density $U = h(y)$ for an increasing or decreasing function $h(y)$.
3. **The method of moment generating functions:** This method is based on the uniqueness theorem for moment generating functions. That is, if two random variables have identical moment generating functions, the two random variables possess the same probability distribution.

The Method of Distribution Functions

Summary of the Distribution Function Method

Let U be a function of the random variables $Y_1, Y_2, \ldots, Y_n$.

1. Find the region $U = u$ in the $(y_1, y_2, \ldots, y_n)$
2. Find the region $U \leq u$.
3. Find $F_U(u) = P(U \leq u)$ by integrating $f(y_1, y_2, \ldots, y_n)$ over the region $U \leq u$.
4. Find the density function $f_U(u)$ by differentiating $F_U(u)$. Thus $f_U(u) = \frac{dF_U(u)}{du}$.

2.2. Example 1. Let the probability density function of X be given by

$$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0 & otherwise \end{cases} \quad (3)$$

Now find the probability density of $Y = X^3$.

Let G(y) denote the value of the distribution function of Y at y and write

$$\begin{aligned} G(y) &= P(Y \leq y) \\ &= P(X^3 \leq y) \qquad (4) \\ &= P\left(X \leq y^{1/3}\right) \\ &= \int_0^{y^{1/3}} 6x(1-x)\,dx \\ &= \int_0^{y^{1/3}} \left(6x - 6x^2\right)dx \\ &= \left(3x^2 - 2x^3\right)\Big|_0^{y^{1/3}} \\ &= 3y^{2/3} - 2y \end{aligned}$$

Now differentiate G(y) to obtain the density function g(y)

$$\begin{aligned} g(y) &= \frac{dG(y)}{dy} \\ &= \frac{d}{dy}\left(3y^{2/3} - 2y\right) \end{aligned}$$

$$= 2y^{-1/3} - 2 \quad (5)$$

$$= 2\left(y^{-1/3} - 1\right), 0 < y < 1$$

2.3. **Example 2.** Let the probability density function of x_1 and of x_2 be given by

$$f(x_1, x_2) = \begin{cases} 2e^{-x_1 - 2x_2}, x_1 > 0, x_2 > 0 \\ 0 \; otherwise \end{cases} \quad (6)$$

Now find the probability density of $Y = X_1 + X_2$. Given that Y is a linear function of X_1 and X_2, we can easily find F(y) as follows.

Let $F_Y(y)$ denote the value of the distribution function of Y at y and write

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= \int_0^y \int_0^{y-x_2} 2e^{-x_1 - 2x_2}\, dx_1\, dx_2 \\ &= \int_0^y -2e^{-x_1 - 2x_2} |_0^{y-x_2}\, dx_2 \quad (7) \\ &= \int_0^y [(-2e^{-y + x_2 - 2x_2}) - (-2e^{-2x2})]\, dx_2 \\ &= \int_0^y -2e^{-y - x_2} + 2e^{-2x2}\, dx_2 \\ &= \int_0^y 2e^{-2x2} - 2e^{-y - x_2}\, dx_2 \end{aligned}$$

Now integrate with respect to x_2 as follows

$$\begin{aligned} F(y) &= P(Y \le y) \\ &= \int_0^y 2e^{-2x2} - 2e^{-y - x_2}\, dx_2 \\ &= -e^{-2x2} + 2e^{-y - x_2} |_0^y \quad (8) \\ &= -e^{-2y} + 2e^{-y-y} - \left[-e^0 + 2e^{-y}\right] \\ &= e^{-2y} - 2e^{-y} + 1 \end{aligned}$$

Now differentiate $F_Y(y)$ to obtain the density function f(y)

$$\begin{aligned} F_Y(y) &= \frac{dF(y)}{dy} \\ &= \frac{d}{dy}\left(e^{-2y} - 2e^{-y} + 1\right) \\ &= -2e^{-2y} + 2e^{-y} \\ &= 2e^{-2y}\left(-1 + e^{y}\right) \end{aligned} \tag{9}$$

2.4. Example 3. Let the probability density function of X be given by

$$\begin{aligned} f_X(x) &= \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \quad -\infty < x < \infty \end{aligned} \tag{10}$$

Now let $Y = \Phi(X) = e^X$. We can then find the distribution of Y by integrating the density function of X over the appropriate area that is defined as a function of y. Let $F_Y(y)$ denote the value of the distribution function of Y at y and write

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P\left(e^X \le y\right) = P(X \le \ln y), \; y > 0 \\ &= \int_{-\infty}^{\ln y} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx, \; y > 0 \end{aligned} \tag{11}$$

Now differentiate $F_Y(y)$ to obtain the density function f(y). In this case we will need the rules for differentiating under the integral sign. They are given by theorem 2 which we state below without proof.

Theorem 2. *Suppose that f and $\frac{\partial f}{\partial x}$ are continuous in the rectangle*

$$R = \{(x, t) : a \le x \le b, c \le t \le d\}$$

and suppose that $u_0(x)$ and $u_1(x)$ are continuously differentiable for $a \le x \le b$ with the range of $u_0(x)$ and $u_1(x)$ in (c, d). If ψ is given by

$$\psi(x) = \int_{u_0(x)}^{u_1(x)} f(x, t)\, dt \tag{12}$$

then

$$\begin{aligned} \frac{d\psi}{dx} &= \frac{\partial}{\partial x}\int_{u_0(x)}^{u_1(x)} f(x, t)\, dt \\ &= f(x, u_1(x))\frac{du_1(x)}{dx} - f(x, u_0(x))\frac{du_0}{dx} + \int_{u_0(x)}^{u_1(x)} \frac{\partial f(x, t)}{\partial x}\, dt \end{aligned} \tag{13}$$

If one of the bounds of integration does not depend on x, then the term involving its derivative will be zero.

For a proof of theorem 2 see (Protter [3, p. 425]). Applying this to equation 11 we obtain

$$
\begin{aligned}
F_Y(y) &= \int_{-\infty}^{\ln y} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx,\ y>0 \\
F'_Y(y) &= f_Y(y) = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{(\ln y-\mu)^2}{2\sigma^2}\right]\left(\frac{1}{y}\right)\right) \\
&\quad + \int_{-\infty}^{\ln y} \frac{d}{dy}\left(\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\right) dx \\
&= \left(\frac{1}{y\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{(\ln y-\mu)^2}{2\sigma^2}\right]\right)
\end{aligned}
\qquad (14)
$$

Bivariate transformation

Distribution-Function Technique

follows essentially the same pattern as the univariate case:

The new random variable Y is now defined in terms of two 'old' RVs, say X_1 and X_2, by $y \equiv g(X_1, X_2)$. We find $F_Y(y) = \Pr(Y < y) = \Pr[g(X_1, X_2) < y]$ by realizing that the $g(X_1, X_2) < y$ inequality (for X_1 and X_2, y is considered fixed) will now result in some *2-D region*, and then integrating $f(x_1, x_2)$ over this region.

Thus, the technique is simple in principle, but often quite involved in terms of technical details.

EXAMPLES:

1. Suppose that X_1 and X_2 are independent RVs, both from $\mathcal{E}(1)$, and $Y = \frac{X_2}{X_1}$.

   Solution: $F_Y(y) = \Pr\left(\frac{X_2}{X_1} < y\right) = \Pr(X_2 < yX_1) = \iint\limits_{0<x_2<yx_1} e^{-x_1-x_2}\, dx_1\, dx_2 = \int\limits_0^\infty e^{-x_1} \int\limits_0^{yx_1} e^{-x_2}\, dx_2\, dx_1 = \int\limits_0^\infty e^{-x_1}(1-e^{-yx_1})\, dx_1 = \int\limits_0^\infty (e^{-x_1} - e^{-x_1(1+y)})\, dx_1 =$

   $1 - \frac{1}{1+y}$, where $y > 0$. This implies that $f_Y(y) = \frac{1}{(1+y)^2}$ when $y > 0$. (The median $\tilde{\mu}$ of this distribution equals to 1, the lower and upper quartiles are $Q_L = \frac{1}{3}$ and $Q_U = 3$).

2. This time Z_1 and Z_2 are independent RVs from $\mathcal{N}(0,1)$ and $Y = Z_1^2 + Z_2^2$ [here, we know the answer: χ_2^2, let us proceed anyhow].

   Solution: $F_Y(y) = \Pr(Z_1^2 + Z_2^2 < y) = \frac{1}{2\pi} \iint\limits_{z_1^2+z_2^2<y} e^{-\frac{z_1^2+z_2^2}{2}}\, dz_1\, dz_2 = \frac{1}{2\pi} \int\limits_0^{2\pi} \int\limits_0^{\sqrt{y}} e^{-\frac{r^2}{2}} \cdot r\, dr\, d\theta =$[substitution: $w = \frac{r^2}{2}$] $\int\limits_0^{\frac{y}{2}} e^{-w}\, dw = 1 - e^{-\frac{y}{2}}$ where (obviously) $y > 0$.

   This is the *exponential* distribution with $\beta = 2$ [not χ_2^2 as expected, how come?]. It does not take long to realize that the two distributions are identical.

Method of Moment Generating Functions

Theorem (6.1 - Uniqueness of Moment Generating Functions)

Let $m_X(t)$ and $m_Y(t)$ denote the moment generating functions of random variables X and Y, respectively. If both moment generating functions exist and $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.

EXAMPLE 6 Suppose X has a normal distribution with mean 0 and variance 1. Let $Y = X^2$, and find the distribution of Y.

$$m_Y(t) = \mathscr{E}[e^{tY}] = \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2(1-2t)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{(1-2t)^{-\frac{1}{2}}}{(1-2t)^{-\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2(1-2t)}\,dx$$

$$= (1-2t)^{-\frac{1}{2}} = \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{\frac{1}{2}} \quad \text{for} \quad t < \frac{1}{2},$$

which we recognize as the moment generating function of a gamma with parameters $r = \frac{1}{2}$ and $\lambda = \frac{1}{2}$. (It is also called a chi-square distribution with one degree of freedom. See Subsec. 4.3 of Chap. VI.) ////

EXAMPLE 7 Let X_1 and X_2 be two independent standard normal random variables. Let $Y_1 = g_1(X_1, X_2) = X_1 + X_2$ and $Y_2 = g_2(X_1, X_2) = X_2 - X_1$. Find the joint distribution of Y_1 and Y_2.

$$\begin{aligned} m_{Y_1, Y_2}(t_1, t_2) &= \mathscr{E}[e^{Y_1 t_1 + Y_2 t_2}] \\ &= \mathscr{E}[e^{(X_1+X_2)t_1 + (X_2 - X_1)t_2}] \\ &= \mathscr{E}[e^{X_1(t_1 - t_2) + X_2(t_1+t_2)}] \\ &= \mathscr{E}[e^{X_1(t_1-t_2)}]\mathscr{E}[e^{X_2(t_1+t_2)}] \\ &= m_{X_1}(t_1 - t_2) m_{X_2}(t_1 + t_2) \\ &= \exp\frac{(t_1 - t_2)^2}{2} \exp\frac{(t_1+t_2)^2}{2} \\ &= \exp(t_1^2 + t_2^2) = \exp\frac{2t_1^2}{2} \exp\frac{2t_2^2}{2} \\ &= m_{Y_1}(t_1) m_{Y_2}(t_2). \end{aligned}$$

We note that Y_1 and Y_2 are independent random variables(
and each has a normal distribution with mean 0 and variance 2.

4.2 Distribution of Sums of Independent Random Variables

In this subsection we employ the moment-generating-function technique to find the distribution of the sum of independent random variables.

Theorem 9 If $X_1, \ldots, X_n$ are independent random variables and the moment generating function of each exists for all $-h < t < h$ for some $h > 0$, let $Y = \sum_1^n X_i$; then

$$m_Y(t) = \mathscr{E}[\exp \sum X_i t] = \prod_{i=1}^n m_{X_i}(t) \qquad \text{for} \quad -h < t < h.$$

PROOF

$$m_Y(t) = \mathscr{E}[\exp \sum X_i t] = \mathscr{E}\left[\prod_{i=1}^n e^{X_i t}\right]$$
$$= \prod_{i=1}^n \mathscr{E}[e^{X_i t}] = \prod_{i=1}^n m_{X_i}(t)$$

using Theorem 9 of Chap. IV. ////

EXAMPLE 9 Suppose that $X_1, \ldots, X_n$ are independent Bernoulli random variables; that is, $P[X_i = 1] = p$, and $P[X_i = 0] = 1 - p$. Now

$$m_{X_i}(t) = pe^t + q.$$

So

$$m_{\sum X_i}(t) = \prod_{i=1}^n m_{X_i}(t) = (pe^t + q)^n,$$

the moment generating function of a binomial random variable; hence $\sum_1^n X_i$ has a binomial distribution with parameters n and p. ////

EXAMPLE 10 Suppose that $X_1, \ldots, X_n$ are independent Poisson distributed random variables, X_i having parameter λ_i. Then

$$m_{X_i}(t) = \mathscr{E}[e^{tX_i}] = \exp \lambda_i(e^t - 1),$$

and hence

$$m_{\sum X_i}(t) = \prod_{i=1}^n m_{X_i}(t) = \prod_{i=1}^n \exp \lambda_i(e^t - 1) = \exp \sum \lambda_i(e^t - 1),$$

EXAMPLE 11 Assume that $X_1, \ldots, X_n$ are independent and identically distributed exponential random variables; then

$$m_{X_i}(t) = \frac{\lambda}{\lambda - t}.$$

So

$$m_{\Sigma X_i}(t) = \prod_{i=1}^{n} m_{X_i}(t) = \left(\frac{\lambda}{\lambda - t}\right)^n,$$

which is the moment generating function of a gamma distribution with parameters n and λ; hence,

$$f_{\Sigma X_i}(x) = \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} I_{(0,\infty)}(x),$$

the density of a gamma distribution with parameters n and λ. ////

EXAMPLE 12 Assume that $X_1, \ldots, X_n$ are independent random variables and

$$X_i \sim N(\mu_i, \sigma_i^2);$$

then

$$a_i X_i \sim N(a_i \mu_i, a_i^2 \sigma_i^2),$$

and

$$m_{a_i X_i}(t) = \exp\left(a_i \mu_i t + \tfrac{1}{2} a_i^2 \sigma_i^2 t^2\right).$$

$$m_{\Sigma a_i X_i}(t) = \prod_{i=1}^{n} m_{a_i X_i}(t) = \exp\left[\left(\sum a_i \mu_i\right)t + \tfrac{1}{2}\left(\sum a_i^2 \sigma_i^2\right)t^2\right],$$

which is the moment generating function of a normal random variable; so

$$\sum_1^n a_i X_i \sim N\left(\sum_1^n a_i \mu_i, \sum_1^n a_i^2 \sigma_i^2\right).$$