Direct Methods for Solving Linear Systems

Linear systems of equations are associated with many problems in engineering and science, as well as with applications of mathematics to the social sciences and the quantitative study of business and economic problems.

In this chapter we consider direct methods for solving a linear system of n equations in n variables. Such a system has the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
. \qquad\qquad\qquad & . \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

In this system we are given the constants a_{ij}, for each i, j = 1, 2, . . . , n, and b_i, for each i = 1, 2, . . . , n, and we need to determine the unknowns $x_1, \ldots, x_n$.

The system can be presented by matrices as follow:

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

or

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

The above matrix is called the Augmented Matrix.

Gauss Elimination Method:

To solve the system of equation we need the following steps:

1- **Forward Elimination:** to find

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22} & \cdots & a_{2n} & b_2 \\ 0 & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} & b_n \end{array}\right]$$

i.e. we get the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\ a''_{33}x_3 + \ldots + a''_{n}x_n &= b''_3 \\ . \qquad & \quad . \\ & . \end{aligned}$$

2- **Backward Substitution:** to find x_i values.

Example: Find the solution of the system of equations:

$$
\begin{aligned}
E_1 &: \quad x_1 + x_2 \qquad\quad + 3x_4 = 4\\
E_2 &: \quad 2x_1 + x_2 - x_3 + x_4 = 1\\
E_3 &: \quad 3x_1 - x_2 - x_3 + 2x_4 = -3\\
E_4 &: \quad -x_1 + 2x_2 + 3x_3 - x_4 = 4
\end{aligned}
$$

Solution:

1- Forward Elimination:

We eliminate x_1 from E_2, E_3 and E_4 by the following steps:

$$
\begin{aligned}
(E_2 - 2E_1) &\rightarrow (E_2)\\
(E_3 - 3E_1) &\rightarrow (E_3)\\
(E_4 + E_1) &\rightarrow (E_4)
\end{aligned}
$$

So we get

$$
\begin{aligned}
E_1 &: \quad x_1 + x_2 \qquad\quad + 3x_4 = 4\\
E_2 &: \quad\quad - x_2 - x_3 - 5x_4 = -7\\
E_3 &: \quad\quad - 4x_2 - x_3 - 7x_4 = -15\\
E_4 &: \quad\quad 3x_2 + 3x_3 + 2x_4 = 8
\end{aligned}
$$

Now, we eliminate x_2 from E_3 and E_4 as follow

$$(E_3 - 4E_2) \rightarrow (E_3)$$

$$(E_4 + 3E_2) \rightarrow (E_4)$$

So we get

$$\begin{aligned}
E_1: &\quad x_1 + x_2 \qquad\quad + \ 3x_4 = 4, \\
E_2: &\quad\ \ - x_2 - \ x_3 - \ 5x_4 = -7, \\
E_3: &\qquad\qquad\ \ 3x_3 + 13x_4 = 13, \\
E_4: &\qquad\qquad\qquad\ - 13x_4 = -13.
\end{aligned}$$

2- **Backward Substitution**:

From E_4: $x_4=1$

Put x_4 in E_3 we get: $x_3= 0$

Put x_3 and x_4 in E_2 we get: $x_2=2$

Put x_2, x_3 and x_4 in E_1 we get: $x_1=-1$

Example2:

Use Gaussian Elimination Method to find the solution of the linear system

$E1: x_1 - x_2 + 2x_3 - x_4 = -8,$

$E2: 2x_1 - 2x_2 + 3x_3 - 3x_4 = -20,$

$E3: x_1 + x_2 + x_3 = -2,$

$E_4: \ x_1 - x_2 + 4x_3 + 3x_4 = 4,$

Solution:

1- Forward Elemination:

We eliminate x_1 from E_2, E_3, and E_4 as follow

$(E_2 - 2E_1) \rightarrow (E_2)$, $(E_3 - E_1) \rightarrow (E_3)$, and $(E_4 - E_1) \rightarrow (E_4)$,

We get

$E_1: x_1 - x_2 + 2x_3 - x_4 = -8,$

$E_2: \quad -x_3 - x_4 = -4,$

$E_3: \quad 2x_2 - x_3 + x_4 = 6,$

$E_4: \quad 2x_3 + 4x_4 = 12,$

In E_2, note that the coefficient of x_2 is zero, so we replace E_2 and E_3 by each other

$E_1: x_1 - x_2 + 2x_3 - x_4 = -8,$

$E_2: \quad 2x_2 - x_3 + x_4 = 6,$

$E_3: \quad -x_3 - x_4 = -4,$

$E_4: \quad 2x_3 + 4x_4 = 12,$

Now, we need only to eliminate x_3 from E_4 as follow: $(E_4 + 2E_3) \rightarrow (E_4)$,

We get

$E_1 : x_1 - x_2 + 2x_3 - x_4 = -8,$

$E_2 : \quad 2x_2 - x_3 + x_4 = 6,$

$E_3 : \quad -x_3 - x_4 = -4,$

$E_4 : \quad 2x_4 = 4,$

2- Backward Subtraction:

From E_4, $x_4=2$, from E_3, $x_3= 2$, from E_2, $x_2=3$, from E_1, $x_1=7$.

Exercises:

Use the Gaussian Elimination Method to solve the following linear systems,

a. $2x_1 - 1.5x_2 + 3x_3 = 1,$

$-x_1 + 2x_3 = 3,$

$4x_1 - 4.5x_2 + 5x_3 = 1.$

b. $x_1 + x_2 + x_4 = 2,$

$2x_1 + x_2 - x_3 + x_4 = 1,$

$4x_1 - x_2 - 2x_3 + 2x_4 = 0,$

$3x_1 - x_2 - x_3 + 2x_4 = -3.$