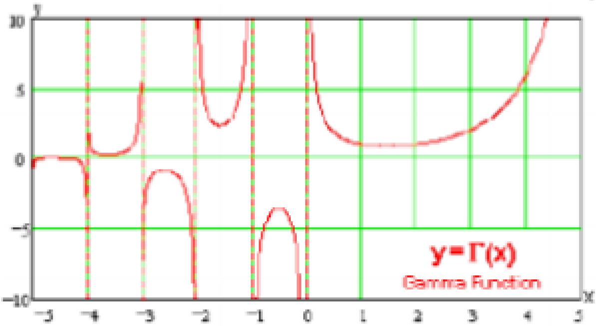
Gamma and Beta functions

The Gamma function is defined by the integral

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \qquad ; n > 0$$



The Gamma function satisfies the recursive properties:

1. $\Gamma(n+1) = n\Gamma(n) \qquad \forall\, n \neq 0\ , n \notin \mathbb{Z}^-$
2. $\Gamma(n+1) = n! \qquad n \in \mathbb{N}$
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
4. $\Gamma(p)\Gamma(1-p) = \dfrac{\pi}{\sin \pi p} \quad ; \ 0 < p < 1$

Example 1: Find 1. $\Gamma\left(\frac{3}{2}\right)$ 2. $\Gamma\left(\frac{5}{2}\right)$ 3. $\Gamma\left(-\frac{1}{2}\right)$ 4. $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

1. $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

2. $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \times \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$

3. $\Gamma(n+1) = n\Gamma(n) \quad \Rightarrow \quad \Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\,\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

4. $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \dfrac{\pi}{\sin\frac{\pi}{4}} = \dfrac{\pi}{1/\sqrt{2}} = \sqrt{2}\,\pi$

Example 2: Evaluate each of the following integrals

1. $$\int_0^\infty x\sqrt{x}\, e^{-x}dx = \int_0^\infty x^{\frac{3}{2}} e^{-x}dx = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\times\frac{1}{2}\sqrt{\pi} = \frac{3}{4}\sqrt{\pi}$$

2. $$\int_0^\infty \frac{e^{-y^2}}{y^2}dy$$

Let $x = y^2 \quad \Rightarrow \quad dx = 2ydy \quad \Rightarrow \quad dy = \dfrac{dx}{2\sqrt{x}}$

$$\int_0^\infty \frac{e^{-y^2}}{y^2}dy = \int_0^\infty \frac{e^{-x}}{x}\frac{dx}{2\sqrt{x}} = \frac{1}{2}\int_0^\infty x^{-\frac{3}{2}} e^{-x}dx$$

$$= \frac{1}{2}\Gamma\left(-\frac{3}{2}+1\right) = \frac{1}{2}\Gamma\left(-\frac{1}{2}\right) = -\sqrt{\pi}$$

3. $$\int_0^\infty \sqrt{t}\, e^{-\sqrt[3]{t}}dt$$

Let $x = \sqrt[3]{t} \quad \Rightarrow \quad dx = \dfrac{dt}{3\sqrt[3]{t^2}} \quad \Rightarrow \quad dt = 3x^2dx$

$$\int_0^\infty \sqrt{t}\, e^{-\sqrt[3]{t}}dt = \int_0^\infty x^{\frac{3}{2}} e^{-x}3x^2dx = 3\int_0^\infty x^{\frac{7}{2}} e^{-x}dx = 3\Gamma\left(\frac{9}{2}\right)$$

$$= 3\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi} = \frac{315}{16}\sqrt{\pi}$$

4. $$\int_3^\infty x^2e^{-x}dx$$

Let $u = x-3$ *then* $du = dx, \quad x = 3 \Rightarrow u = 0$

$$\int_0^\infty (u+3)^2e^{-u-3}du = \int_0^\infty (u^2+6u+9)e^{-u}du$$

$$= \int_0^\infty u^2e^{-u}du + \int_0^\infty 6u\, e^{-u}du + \int_0^\infty 9\, e^{-u}du$$

$$= \Gamma(3) + 6\Gamma(2) + 9\Gamma(1)$$
$$= 2! + 6\times 1! + 9\times 0! = 2+6+9 = 17$$

Beta function

The Beta function is defined by the integral

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ; \; n > 0 \;, \qquad m > 0$$

The Beta function satisfies the recursive properties:

1. The Beta function is symmetric that is : $B(m,n) = B(n,m)$

2. $$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Example 3: Evaluate 1. $B(3,4)$ 2. $B\left(\frac{1}{2},\frac{5}{2}\right)$

1. $$B(3,4) = \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{2!\times 3!}{6!} = \frac{2\times 3!}{6\times 5\times 4\times 3!} = \frac{1}{60}$$

2. $$B\left(\frac{1}{2},\frac{5}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{5}{2}\right)} = \frac{\sqrt{\pi}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{\Gamma(3)} = \frac{3\pi}{8}$$

Example 4: Evaluate each of the following integrals

1. $$\int_0^1 x^3(1-x)^4\,dx = B(4,5) = \frac{\Gamma(4)\Gamma(5)}{\Gamma(9)} = \frac{3!\times 4!}{8!} = \frac{6\times 4!}{8\times 7\times 6\times 5\times 4!} = \frac{1}{280}$$

2. $$\int_0^2 \frac{x^2}{\sqrt{2-x}}dx$$

Let $x = 2y \quad \Rightarrow \quad dx = 2dy \;, \qquad x = 0 \Rightarrow y = 0$ and $x = 2 \Rightarrow y = 1$

$$\int_0^2 \frac{x^2}{\sqrt{2-x}}dx = \int_0^1 \frac{4y^2}{\sqrt{2-2y}}2dy = \frac{8}{\sqrt{2}}\int_0^1 \frac{y^2}{\sqrt{1-y}}dy$$

$$= 4\sqrt{2}\int_0^1 y^2(1-y)^{-1/2}\,dy$$

$$= 4\sqrt{2}\,B\left(3,\frac{1}{2}\right) = 4\sqrt{2}\frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = \frac{64\sqrt{2}}{15}$$

3. $\int_0^2 t^4\sqrt{4-t^2}\,dt$

Let $t^2 = 4x \quad \Rightarrow \quad 2tdt = 4dx \;, \qquad t=0 \Rightarrow x=0$ and $t=2 \Rightarrow x=1$

$$\int_0^2 t^4\sqrt{4-t^2}\,dt = \int_0^2 t^2 \times t\sqrt{4-t^2}\,\times tdt = \int_0^1 4x \times 2\sqrt{x}\sqrt{4-4x}\,\times 2dx$$

$$= 32\int_0^1 x^{3/2}\sqrt{1-x}\,dx = 32B\left(\frac{5}{2},\frac{3}{2}\right)$$

$$= \frac{32\,\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)} = \frac{32 \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{3!} = 2\pi$$

Many integrals can be expressed through beta and gamma functions. Two of special interest are:

1. $$\int_0^\infty \frac{x^{p-1}}{1+x}dx = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi} \qquad ; \qquad 0 < p < 1$$

2. $$\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = \frac{1}{2}B(m,n)$$

Example 5 : Evaluate each of the following integrals

1. $$\int_0^\infty \frac{x^{-2/3}}{1+x}dx = \int_0^\infty \frac{x^{(1/3)-1}}{1+x}dx = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$$

2. $$\int_0^{\pi/2} \sin^9\theta\cos^5\theta\,d\theta$$

$2m-1=9 \quad \Rightarrow \quad m=5$ and $2n-1=5 \quad \Rightarrow \quad n=3$

$$\int_0^{\pi/2} \sin^9\theta\cos^5\theta\,d\theta = \frac{1}{2}B(5,3) = \frac{\Gamma(5)\Gamma(3)}{2\Gamma(5+3)} = \frac{4! \times 2!}{2 \times 7!} = \frac{1}{210}$$

3. $\int_0^{\pi/2} \sin^5 x \, dx$

$$2m-1=5 \;\Rightarrow\; m=3 \quad \text{and} \quad 2n-1=0 \;\Rightarrow\; n=\frac{1}{2}$$

$$\int_0^{\pi/2} \sin^5 x \, dx = \frac{1}{2}B\left(3,\frac{1}{2}\right) = \frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{7}{2}\right)} = \frac{2!\sqrt{\pi}}{2\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}} = \frac{8}{15}$$

Exercises

Evaluate each of the following integrals

1. $\int_0^{\infty} x^6 \, e^{-3x} dx$
2. $\int_0^{\infty} \sqrt[4]{t} \, e^{-\sqrt{t}} \, dt$
3. $\int_0^{1} x^5 (1-x)^6 \, dx$
4. $\int_0^{\pi/2} \cos^4 x \, dx$
5. $\int_0^{\pi/2} \sin^5 x \cos^4 x \, dx$
6. $\int_0^{\infty} \frac{1}{\sqrt[4]{x}\,(1+x)} dx$
7. $\int_0^{1} x^4 \left(1-\sqrt{x}\right)^5 dx$