1.2: The Scalar Product

Given two vectors $\vec{A}$ and $\vec{B}$, the scalar product or "dot" product $\vec{A}\cdot\vec{B}$, is the scalar defined by the equation

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

It follows from the above definition that scalar multiplication is commutative

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

since $A_x = B_x$, $A_y = B_y$ and $A_z = B_z$

It follows that it is distributive

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

the angle between $\vec{A}$ and $\vec{B}$ is given by

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$$

If $\vec{A}\cdot\vec{B}$ is equal to zero and neither $\vec{A}$ nor $\vec{B}$ is null, then $\cos\theta$ is zero and $\vec{A}\perp\vec{B}$.

$$A^2 = |\vec{A}|^2 = \vec{A}\cdot\vec{A}$$

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$

$$\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$$

Ex3:- Find the cosine of the angle between $\vec{A}=[1,1,1]$ and $\vec{B}=[1,1,0]$

Solution

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|A||B|} = \frac{1+1+0}{\sqrt{3}\,\sqrt{2}} = \sqrt{\frac{2}{3}} = 0.8165$$

Ex:- The vector $a\hat{i}+\hat{j}-\hat{k}$ is perpendicular to the vector $\hat{i}+2\hat{j}-3\hat{k}$, what is the value of a?

Solution

If the vectors are perpendicular to each other, their dot product must vanish (cos 90 = zero).

hence, we have

$$(a\hat{i}+\hat{j}-\hat{k})\cdot(\hat{i}+2\hat{j}-3\hat{k}) = a+2+3 = a+5 = 0$$

hence $a=-5$

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1.3: The Vector Product

Given two vectors $\vec{A}$ and $\vec{B}$, the vector product or cross product, $\vec{A} \times \vec{B}$, is defined as the vector whose components are given by the equation

$$\vec{A} \times \vec{B} = [A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x]$$

It can be shown that the following rules hold for cross multiplications:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

$$n(\vec{A} \times \vec{B}) = (n\vec{A}) \times \vec{B} = \vec{A} \times (n\vec{B})$$

also

$$\left.\begin{aligned} \hat{i} \times \hat{i} &= \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \\ \hat{j} \times \hat{k} &= \hat{i} = -\hat{k} \times \hat{j} \\ \hat{i} \times \hat{j} &= \hat{k} = -\hat{j} \times \hat{i} \\ \hat{k} \times \hat{i} &= \hat{j} = -\hat{i} \times \hat{k} \end{aligned}\right\} \text{Prove it (H.W)}$$

For example

$$\hat{i} \times \hat{j} = [0-0, 0-0, 1-0] = [0,0,1] = \hat{k}$$

The cross product expressed in $\hat{i}\hat{j}\hat{k}$ form is

$$\vec{A} \times \vec{B} = \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z) + \hat{k}(A_x B_y - A_y B_x)$$

$$= \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + \hat{j}\begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

and finally

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Let us calculate the magnitude of the cross product. We have

$$|A \times B|^2 = (A_y B_z - A_z B_y)^2 + (A_z B_x - A_x B_z)^2 + (A_x B_y - A_y B_x)^2$$

$$= (A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2) - (A_x B_x + A_y B_y + A_z B_z)^2$$ How

$$= A^2 B^2 - (\vec{A} \cdot \vec{B})^2$$

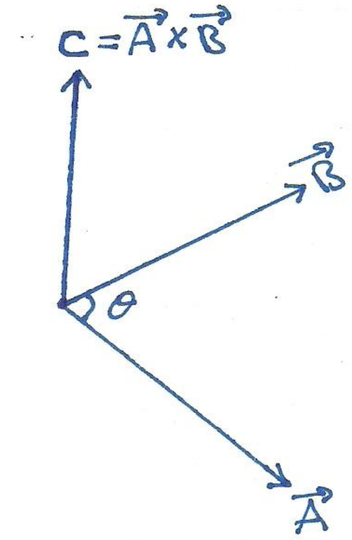
$$|A \times B| = \sqrt{A^2 B^2 - (\vec{A} \cdot \vec{B})^2}$$

$$|\vec{A}\times\vec{B}| = \sqrt{A^2B^2 - A^2B^2\cos^2\theta} = AB\sqrt{1-\cos^2\theta}$$

$$= AB\sin\theta$$

$$\therefore \boxed{\vec{A}\times B = (AB\sin\theta)\,\hat{n}}$$

where $\hat{n}$ is a unit vector normal to the plane of the two vectors $\vec{A}$ and $\vec{B}$



Ex4 :- Given the two vectors $\vec{A} = 2\hat{i}+\hat{j}-\hat{k}$, $\vec{B} = \hat{i}-\hat{j}+2\hat{k}$, find ① $\vec{A}\times\vec{B}$ ② unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$.

solution :

① $$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i}(2-1) + \hat{j}(-1-4) + \hat{k}(-2-1)$$

$$= \hat{i} - 5\hat{j} - 3\hat{k}$$

② $$\hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \frac{\hat{i}-5\hat{j}-3\hat{k}}{[1^2+5^2+3^2]^{1/2}} = \frac{\hat{i}}{\sqrt{35}} - \frac{5\hat{j}}{\sqrt{35}} - \frac{3\hat{k}}{\sqrt{35}}$$