## Hydrostatic Forces in Layered Fluids

All of the above results which employ the linear hydrostatic variation of pressure are valid only for homogeneous fluids. If the fluid is heterogeneous, consisting of individual layers each of constant ysesldensity, then the pressure varies linearly with a different slope in each layer and the preceding ana .must be remedied by computing and summing the separate contributions to the forces and moments

## Buoyancy

The same principles used above to compute hydrostatic forces can be used to calculate the net pressure force acting on completely submerged or floating bodies. These laws of buoyancy, the principles of Archimedes, are that:

1. A completely submerged body experiences a vertical upward force equal to the weight of the displaced fluid; and
2. A floating or partially submerged body displaces its own weight in the fluid in which it floats (i.e., the vertical upward force is equal to the body weight).

The line of action of the buoyancy force in both (1) and (2) passes through the centroid of the displaced volume of fluid; this point is called the *center of buoyancy.* (This point need not correspond to the center of mass of the body, which could have nonuniform density. In the above it has been assumed that the displaced fluid has a constant $\gamma$. If this is not the case, such as in a layered fluid, the magnitude of the buoyant force is still equal to the weight of the displaced fluid, but the line of action of this force passes through the center of gravity of the displaced volume, not the centroid.)

If a body has a weight exactly equal to that of the volume of fluid it displaces, it is said to be *neutrally buoyant* and will remain at rest at any point where it is immersed in a (homogeneous) fluid.

## Stability of Submerged and Floating Bodies

### Submerged Body

A body is said to be in stable equilibrium if, when given a slight displacement from the equilibrium position, the forces thereby created tend to restore it back to its original position. The forces acting on a submerged body are the buoyancy force, $F_B$, acting through the center of buoyancy, denoted by CB, and the weight of the body, $W$, acting through the center of gravity denoted by CG (see Figure 3.1.4). We see from Figure 3.1.4 that if the CB lies above the CG a rotation from the equilibrium position creates a restoring couple which will rotate the body back to its original position — thus, this is a *stable* equilibrium situation. The reader will readily verify that when the CB lies below the CG, the couple that results from a rotation from the vertical increases the displacement from the equilibrium position — thus, this is an *unstable* equilibrium situation.

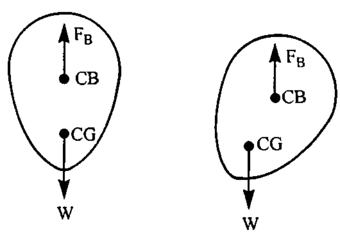


**FIGURE 3.1.4** Stability for a submerged body.

### Partially Submerged Body

The stability problem is more complicated for floating bodies because as the body rotates the location of the center of buoyancy may change. To determine stability in these problems requires that we determine the location of the *metacenter*. This is done for a symmetric body by tilting the body through a small angle $\Delta\theta$ from its equilibrium position and calculating the new location of the center of buoyancy CB′; the point of intersection of a vertical line drawn upward from CB′ with the line of symmetry of the floating body is the metacenter, denoted by $M$ in Figure 3.1.5, and it is independent of $\Delta\theta$ for small angles. If $M$ lies above the CG of the body, we see from Figure 3.1.5 that rotation of the body leads to

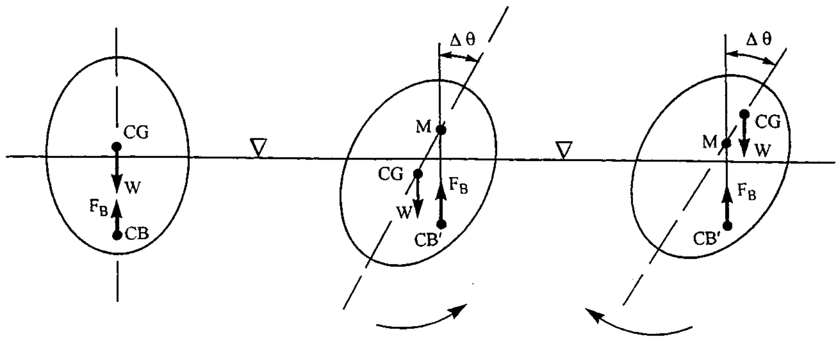


**FIGURE 3.1.5** Stability for a partially submerged body.

a restoring couple, whereas *M* lying below the CG leads to a couple which will increase the displacement. Thus, the stability of the equilibrium depends on whether *M* lies above or below the CG. The directed distance from CG to *M* is called the *metacentric height*, so equivalently the equilibrium is stable if this vector is positive and unstable if it is negative; stability increases as the metacentric height increases. For geometrically complex bodies, such as ships, the computation of the metacenter can be quite complicated.

## Pressure Variation in Rigid-Body Motion of a Fluid

In rigid-body motion of a fluid all the particles translate and rotate as a whole, there is no relative motion between particles, and hence no viscous stresses since these are proportional to velocity gradients. The equation of motion is then a balance among pressure, gravity, and the fluid acceleration, specifically.

$$\nabla p = \rho(\boldsymbol{g} - \boldsymbol{a}) \tag{3.1.18}$$

where $\boldsymbol{a}$ is the uniform acceleration of the body. Equation (3.1.18) shows that the lines of constant pressure, including a free surface if any, are perpendicular to the direction $\boldsymbol{g} - \boldsymbol{a}$. Two important applications of this are to a fluid in uniform linear translation and rigid-body rotation. While such problems are not, strictly speaking, fluid statics problems, their analysis and the resulting pressure variation results are similar to those for static fluids.

### Uniform Linear Acceleration

For a fluid partially filling a large container moving to the right with constant acceleration $\boldsymbol{a} = (a_x, a_y)$ the geometry of Figure 3.1.6 shows that the magnitude of the pressure gradient in the direction $\boldsymbol{n}$ normal to the accelerating free surface, in the direction $\boldsymbol{g} - \boldsymbol{a}$, is

$$\frac{dp}{dn} = \rho\left[a_x^2 + \left(g + a_y\right)^2\right]^{1/2} \tag{3.1.19}$$

and the free surface is oriented at an angle to the horizontal

$$\theta = \tan^{-1}\left(\frac{a_x}{g + a_y}\right) \tag{3.1.20}$$

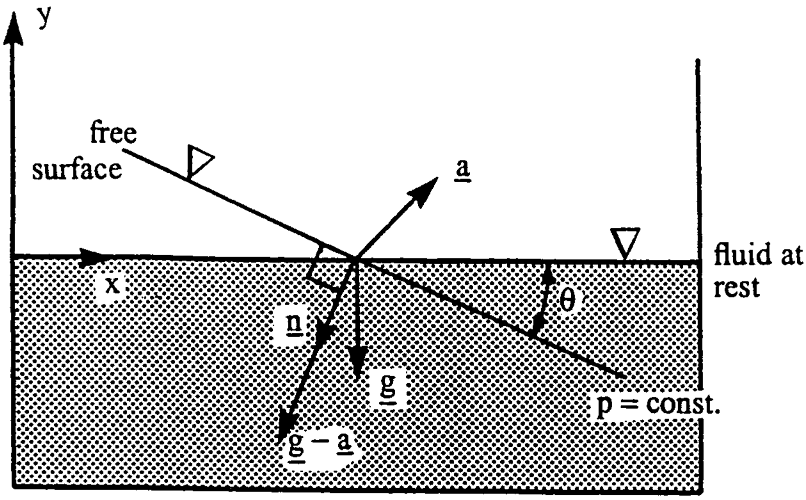


**FIGURE 3.1.6** A fluid with a free surface in uniform linear acceleration.

### Rigid-Body Rotation

Consider the fluid-filled circular cylinder rotating uniformly with angular velocity $\Omega = \Omega \boldsymbol{e}_r$ (Figure 3.1.7). The only acceleration is the centripetal acceleration $\Omega \times \Omega \times \boldsymbol{r}) = -r\Omega^2 \boldsymbol{e}_r$, so Equation 3.1.18 becomes

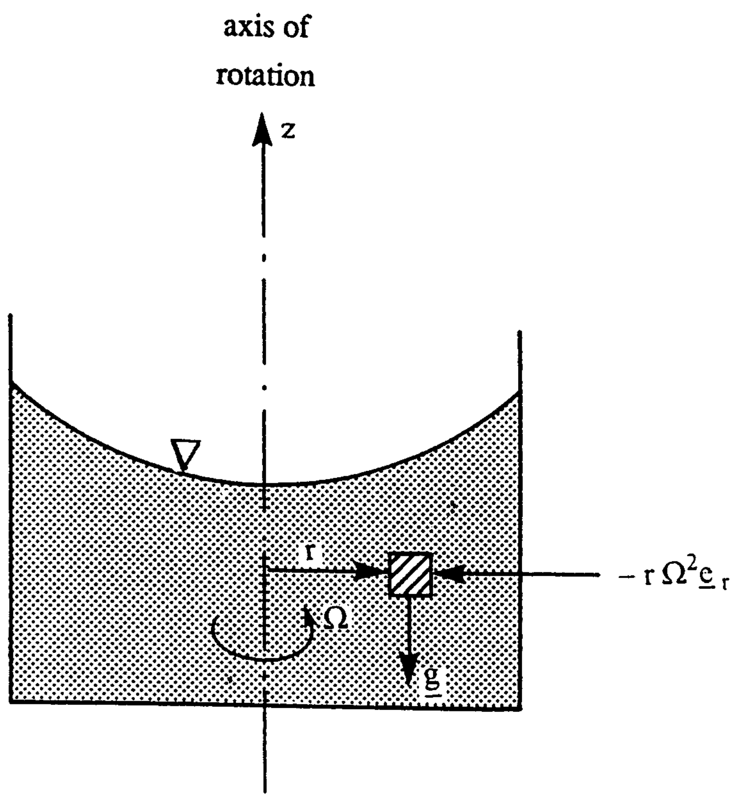


**FIGURE 3.1.7** A fluid with a free surface in rigid-body rotation.

$$\nabla p = \frac{\partial p}{\partial r}\boldsymbol{e}_r + \frac{\partial p}{\partial z}\boldsymbol{e}_z = \rho(\boldsymbol{g} - \boldsymbol{a}) = \rho\left(r\Omega^2\boldsymbol{e}_r - g\boldsymbol{e}_z\right) \tag{3.1.21}$$

or

$$\frac{\partial p}{\partial r} = \rho r \Omega^2, \qquad \frac{\partial p}{\partial z} = -\rho g = -\gamma \tag{3.1.22}$$

Integration of these equations leads to

$$p = p_o - \gamma z + \frac{1}{2}\rho r^2 \Omega^2 \tag{3.1.23}$$

where $p_o$ is some reference pressure. This result shows that at any fixed $r$ the pressure varies hydrostatically in the vertical direction, while the constant pressure surfaces, including the free surface, are paraboloids of revolution.