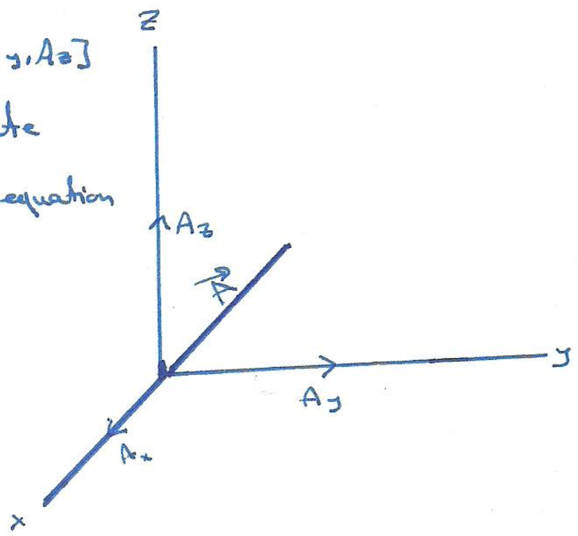
CH.1: Fundamental concepts.

Vectors

1.1. Formal Definitions and Rules of Vector Algebra

A given vector $\vec{A}$ is specified by stating its magnitude and its direction relative to some chosen reference system. A vector is represented diagrammatically by a directed line segment, as shown in figure 1.1.



The component symbol $[A_x, A_y, A_z]$ will be used as an alternate desination of a vector. The equation

$\vec{A} = [A_x, A_y, A_z]$

means that the vector $\vec{A}$ is expressed on the right in terms of its components in a particular coordinate system.

For example, if the vector $\vec{A}$ represents a displacement from point $P_1(x_1, y_1, z_1)$ to the point $P_2(x_2, y_2, z_2)$ then $A_x = x_2 - x_1$, $A_y = y_2 - y_1$, $A_z = z_2 - z_1$.

If $\vec{A}$ represents a force, then A_x is the x component of the force, and so on.

We begin the study of vector algebra with some formal statements concerning vectors:

1- Equality of vectors:-

The equation

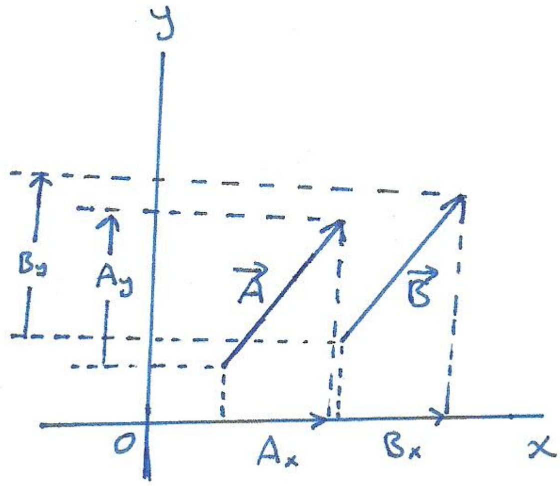
$$\vec{A} = \vec{B}$$

or

$$[A_x, A_y, A_z] = [B_x, B_y, B_z]$$

is equivalent to the three eqs.

$$A_x = B_x \,,\ A_y = B_y \,,\ A_z = B_z$$



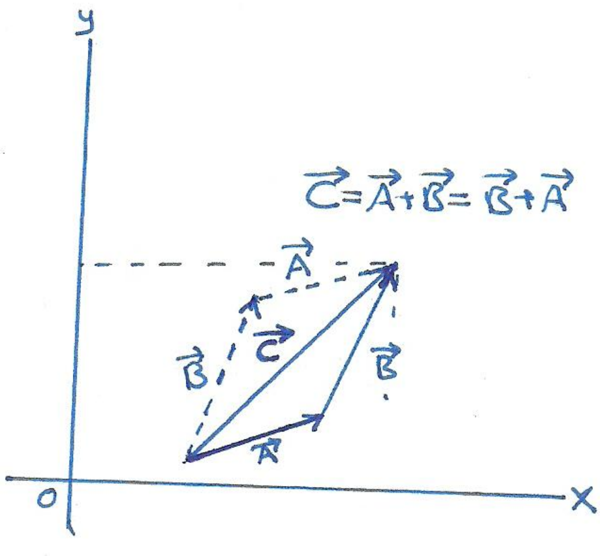
that is, two vectors are equal if & only if, their respective components are equal.

2- Vector addition:-

The addition of two vectors is defined by the eqs.

$$\vec{A} + \vec{B} = [A_x, A_y, A_z] + [B_x, B_y, B_z]$$
$$= [A_x + B_x, A_y + B_y, A_z + B_z]$$

The sum of two vectors is a vector whose components are sums of the components of the given vectors



3- Multiplication by a scalar:

If c is a scalar and $\vec{A}$ is a vector

$$c\vec{A} = c[A_x, A_y, A_z] = [cA_x, cA_y, cA_z] = \vec{A}c$$

the product $c\vec{A}$ is a vector whose components are c times those of $\vec{A}$.

4- Vector Subtraction

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}) = [A_x - B_x, A_y - B_y, A_z - B_z]$$

that is, subtraction of a given vector $\vec{B}$ from the vector $\vec{A}$ is equivalent to adding $-\vec{B}$ to $\vec{A}$.

5- The Null Vector

The vector $0 = [0, 0, 0]$ is called the null vector.

the direction of the null vector is undefined.

6- The commutative law of addition

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

since $A_x + B_x = B_x + A_x$, $A_y + B_y = B_y + A_y$ & $A_z + B_z = B_z + A_z$

7- The Associative law:

$$\vec{A}+(\vec{B}+\vec{C})=[A_x+(B_x+C_x),\ A_y+(B_y+C_y),\ A_z+(B_z+C_z)]$$

$$=[(A_x+B_x)+C_x,\ (A_y+B_y)+C_y,\ (A_z+B_z)+C_z]$$

$$=(\vec{A}+\vec{B})+\vec{C}$$

8- The Distributive law:

if c is a scalar

$$c(\vec{A}+\vec{B})=c[A_x+B_x,\ A_y+B_y,\ A_z+B_z]$$

$$=[c(A_x+B_x),\ c(A_y+B_y),\ c(A_z+B_z)]$$

$$=[cA_x+cB_x,\ cA_y+cB_y,\ cA_z+cB_z]$$

$$=c\vec{A}+c\vec{B}$$

9- Magnitude of a vector:

The magnitude of a vector $\vec{A}$, denoted by $|\vec{A}|$ is defined as the square root of the sum of the squares of the components, namely

$$|\vec{A}|=(A_x^2+A_y^2+A_z^2)^{1/2}$$

10 - Unit Coordinate Vectors :

A unit vector is a vector whose magnitude is unity. Unit vectors are often designated by the symbol e

~~The~~ three unit vectors

$e_x = [1,0,0]$, $e_y = [0,1,0]$, $e_z = [0,0,1]$

are called unit coordinate vectors or basis vectors. In terms of basis vectors, $\vec{A}$ can be expressed as

$$\vec{A} = [A_x, A_y, A_z] = [A_x, 0, 0] + [0, A_y, 0] + [0, 0, A_z]$$

$$= A_x[1,0,0] + A_y[0,1,0] + A_z[0,0,1]$$

$$= e_x A_x + e_y A_y + e_z A_z$$

For cartesian unit vectors

$\hat{i} = e_x \quad \hat{j} = e_y \quad \hat{k} = e_z$

$\therefore \quad \vec{A} = \hat{i} A_x + \hat{j} A_y + \hat{k} A_z$

Ex: ① Find the sum and the magnitude of the two vectors $\vec{A} = [1,0,2]$ and $\vec{B} = [0,1,1]$

Solution

$$\vec{A} + \vec{B} = [1,0,2] + [0,1,1] = [1,1,3]$$

$$|\vec{A} + \vec{B}| = (1+1+9)^{1/2} = \sqrt{11}$$

② Express of the vectors $\vec{A}$ & $\vec{B}$ in terms of $\hat{i}, \hat{j}, \hat{k}$

Solution $\quad \vec{A} = \hat{i} + 0\hat{j} + \hat{k}$, $\vec{B} = 0\hat{i} + \hat{j} + \hat{k}$

Ex 2:- A helicopter flies 100m vertically upward, then 500m horizontally east, then 1000 m horizontally north. How far is it from a second helicopter that starts from the same point rising 200m upward, 100m west, & 500m north.? 6

Solution:

The final position of the first helicopter is expressed vectorially as $\vec{A} = [100, 500, 1000]$ and the second as $\vec{B} = [200, -100, 500]$, in meters. Hence the distance between the final positions is given by the expression

$$|\vec{A} - \vec{B}| = |[(100-200), (500+100), (1000-500)]| m$$

$$= (100^2 + 600^2 + 500^2)^{1/2} m = 787.4 m$$

1.2: The Scalar Product

Given two vectors $\vec{A}$ and $\vec{B}$, the scalar product or "dot" product $\vec{A}\cdot\vec{B}$, is the scalar defined by the equation

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

It follows from the above definition that scalar multiplication is commutative

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

since $A_x = B_x$, $A_y = B_y$ and $A_z = B_z$

It follows that it is distributive

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

the angle between $\vec{A}$ and $\vec{B}$ is given by

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$$

If $\vec{A}\cdot\vec{B}$ is equal to zero and neither $\vec{A}$ nor $\vec{B}$ is null, then $\cos\theta$ is zero and $\vec{A}\perp\vec{B}$.

$$A^2 = |\vec{A}|^2 = \vec{A}\cdot\vec{A}$$

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$

$$\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$$

Ex3:- Find the cosine of the angle between $\vec{A}=[1,1,1]$ and $\vec{B}=[1,1,0]$

Solution

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|A||B|} = \frac{1+1+0}{\sqrt{3}\sqrt{2}} = \sqrt{\frac{2}{3}} = 0.8165$$

Ex:- The vector $a\hat{i}+\hat{j}-\hat{k}$ is perpendicular to the vector $\hat{i}+2\hat{j}-3\hat{k}$, what is the value of a?

Solution

If the vectors are perpendicular to each other, thier dot product must vanish ($\cos 90 =$ zero). hence, we have

$$(a\hat{i}+\hat{j}-\hat{k})\cdot(\hat{i}+2\hat{j}-3\hat{k}) = a+2+3 = a+5 = 0$$

hence $a=-5$

1.3: The Vector Product

Given two vectors $\vec{A}$ and $\vec{B}$, the vector product or cross product, $\vec{A} \times \vec{B}$, is defined as the vector whose components are given by the equation

$$\vec{A} \times \vec{B} = [\, A_y B_z - A_z B_y \,,\, A_z B_x - A_x B_z \,,\, A_x B_y - A_y B_x \,]$$

It can be shown that the following rules hold for cross multiplications:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

$$n(\vec{A} \times \vec{B}) = (n\vec{A}) \times \vec{B} = \vec{A} \times (n\vec{B})$$

also

$$\left.\begin{array}{l} \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \\ \hat{j} \times \hat{k} = \hat{i} = -\hat{k} \times \hat{j} \\ \hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i} \\ \hat{k} \times \hat{i} = \hat{j} = -\hat{i} \times \hat{k} \end{array}\right\} \text{Prove it (H.w)}$$

For example

$$\hat{i} \times \hat{j} = [0-0, 0-0, 1-0] = [0,0,1] = \hat{k}$$

The cross product expressed in $\hat{i}\hat{j}\hat{k}$ form is

$$\vec{A} \times \vec{B} = \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z) + \hat{k}(A_x B_y - A_y B_x)$$

$$= \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + \hat{j}\begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

and finally

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Let us calculate the magnitude of the cross product. We have

$$|A \times B|^2 = (A_y B_z - A_z B_y)^2 + (A_z B_x - A_x B_z)^2 + (A_x B_y - A_y B_x)^2$$

$$= (A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2) - (A_x B_x + A_y B_y + A_z B_z)^2 \quad \text{How}$$

$$= A^2 B^2 - (\vec{A} \cdot \vec{B})^2$$

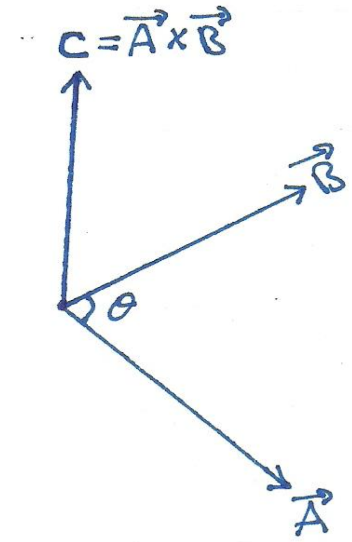
$$|A \times B| = \sqrt{A^2 B^2 - (\vec{A} \cdot \vec{B})^2}$$

$$|\vec{A}\times\vec{B}| = \sqrt{A^2B^2 - A^2B^2\cos^2\theta} = AB\sqrt{1-\cos^2\theta}$$

$$= AB\sin\theta$$

$$\therefore \quad \boxed{\vec{A}\times B = (AB\sin\theta)\,\hat{n}}$$

where $\hat{n}$ is a unit vector normal to the plane of the two vectors $\vec{A}$ and $\vec{B}$



Ex4 :- Given the two vectors
$\vec{A} = 2\hat{i}+\hat{j}-\hat{k}$, $\vec{B} = \hat{i}-\hat{j}+2\hat{k}$, find ① $\vec{A}\times\vec{B}$ ② unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$.

solution :

① $$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i}(2-1) + \hat{j}(-1-4) + \hat{k}(-2-1)$$

$$= \hat{i} - 5\hat{j} - 3\hat{k}$$

② $$\hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \frac{\hat{i}-5\hat{j}-3\hat{k}}{[1^2+5^2+3^2]^{1/2}} = \frac{\hat{i}}{\sqrt{35}} - \frac{5\hat{j}}{\sqrt{35}} - \frac{3\hat{k}}{\sqrt{35}}$$

1.4: Triple Product

The expression

$$\vec{A}\cdot(\vec{B}\times\vec{C})$$

is called the triple scalar product of $\vec{A}, \vec{B}$ & $\vec{C}$

The expression

$$\vec{A}\times(\vec{B}\times\vec{C})$$

is called triple vector product.

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$ H.W

Ex 5:- Given the three vectors $\vec{A} = \hat{i}$, $\vec{B} = \hat{i} - \hat{j}$, and $C = \hat{k}$ find ① $\vec{A}\cdot(\vec{B}\times\vec{C})$ ② $\vec{A}\times(\vec{B}\times\vec{C})$

solution :

① $$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{i}(-1) - \hat{j}(1) + \hat{k}(0)$$

$$= -\hat{i} - \hat{j}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \hat{i}\cdot(-\hat{i} - \hat{j}) = -1$$

② $$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ -1 & -1 & 0 \end{vmatrix} = \hat{i}(0) + \hat{j}(0) + \hat{k}(-1)$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = -\hat{k}$$

or $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})B - (\vec{A}\cdot\vec{B})C = 0(\hat{i} - \hat{j}) - (1-0)\hat{k} = -\hat{k}$