1.6: Position vector of a particle. Velocity and Acceleration in Rectangular Coordinates

The position vector of the particle

$$\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$$

The components of the position vector of a moving particle are functions of the time, namely

$$x = x(t) \qquad y = y(t) \qquad z = z(t)$$

The derivative of $\vec{r}$ with respect to t is called velocity $(\vec{v})$

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\,\dot{x} + \hat{j}\,\dot{y} + \hat{k}\,\dot{z}$$

where

$$\dot{x} = \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt}, \quad \dot{z} = \frac{dz}{dt}$$

The magnitude of the velocity is called speed

$$|\vec{v}| = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}$$

The time derivative of the velocity is called the acceleration $(\vec{a})$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \hat{i}\,\ddot{x} + \hat{j}\,\ddot{y} + \hat{k}\,\ddot{z}$$

Ex 7:- The motion of particle in the xy-plane represented by the equation

$$\vec{r}(t) = \hat{i}\,bt + \hat{j}\left(ct - \frac{gt^2}{2}\right)$$

Find ① velocity ② speed ③ acceleration

Solution

① $\vec{v} = \dfrac{d\vec{r}}{dt} = i\,b + j\,(c - gt)$

② $|\vec{v}| = \sqrt{b^2 + (c-gt)^2}$

③ $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = -\hat{j}\,g$

Ex 8:- suppose the position vector is given by

$$\vec{r} = \hat{i}\,b\sin\omega t + \hat{j}\,b\cos\omega t \qquad (\omega \text{ constant})$$

Find ① velocity ② acceleration

solution

① $\vec{v} = \dfrac{d\vec{r}}{dt} = \hat{i}\,b\omega\cos\omega t - \hat{j}\,b\omega\sin\omega t$

② $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2} = -\hat{i}\,b\omega^2\sin\omega t - \hat{j}\,b\omega^2\cos\omega t$

$$= -\omega^2\underbrace{(\hat{i}\,b\sin\omega t + \hat{j}\,b\cos\omega t)}_{\vec{r}}$$

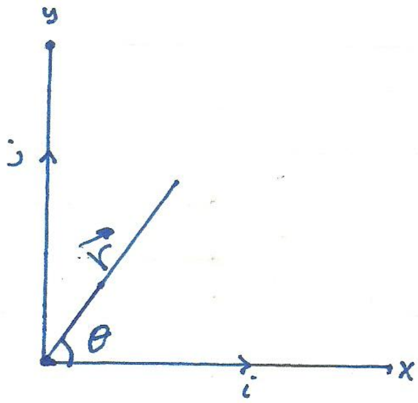
$$\vec{a} = -\omega^2\vec{r}$$

Velocity and Acceleration in Different Coordinates

a) Polar coordinate (r, θ)

The position vector is

$$\vec{r} = \hat{r}\, r \quad \text{---} \; (1)$$



$$V = \frac{d\vec{r}}{dt} = \hat{r}\,\dot{r} + r\,\frac{d\hat{r}}{dt} \quad \text{---} \; (2)$$

From fig. we get

$$x = r\cos\theta \;, \quad y = r\sin\theta \quad \text{--} \; (3)$$

$$\vec{r} = \hat{i}\,x + \hat{j}\,y = \hat{i}\,r\cos\theta + \hat{j}\,r\sin\theta$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\hat{i}\,r\cos\theta + \hat{j}\,r\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = \hat{i}\cos\theta + \hat{j}\sin\theta \quad \text{----} \; (4)$$

$$\frac{d\hat{r}}{d\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta \quad \text{---} \; (5)$$

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{|\partial\vec{r}/\partial\theta|} = \frac{-\hat{i}\,r\sin\theta + \hat{j}\,r\cos\theta}{\sqrt{r^2\sin^2\theta + r^2\cos^2\theta}} = -\hat{i}\sin\theta + \hat{j}\cos\theta \quad \text{---} \; (6)$$

From eq. (5) & eq. (4) we get $\dfrac{d\hat{r}}{d\theta} = \hat{\theta}$

$$\frac{d\hat{\theta}}{d\theta} = -\hat{i}\cos\theta - \hat{j}\sin\theta = -(\hat{i}\cos\theta + \hat{j}\sin\theta) \quad \text{----} \; (7)$$

From eq. (4) and eq. (7) we get $\dfrac{d\hat{\theta}}{d\theta} = -\hat{r}$

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also, by using chain rule

$$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta} \cdot \frac{d\theta}{dt} = \hat{\theta}\,\dot{\theta} \qquad \text{----(8)}$$

&

$$\frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta} \cdot \frac{d\theta}{dt} = -\hat{r}\,\dot{\theta}$$

Now eq. 2 become

$$\boxed{\vec{V} = \hat{r}\,\dot{r} + \hat{\theta}\,r\,\dot{\theta}} \qquad \text{----(9)}$$

(velocity in term of polar coordinate)

where $\dot{r}$ is the radial component of velocity vector (V_r) and $r\dot{\theta}$ is the transverse component (V_θ)

to find the acceleration,

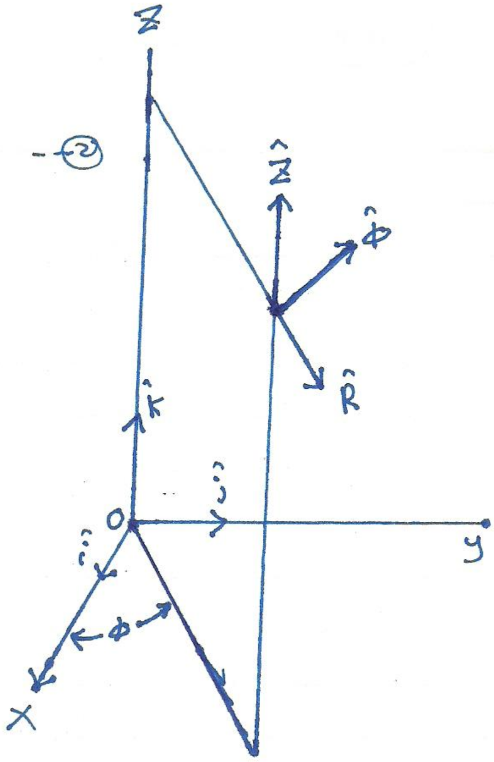
$$\vec{a} = \frac{d\vec{V}}{dt} = \hat{r}\,\ddot{r} + \dot{r}\frac{d\hat{r}}{dt} + \hat{\theta}\,\dot{r}\,\dot{\theta} + \hat{\theta}\,r\,\ddot{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$\boxed{\vec{a} = \hat{r}(\ddot{r} - r\dot{\theta}^2) + \hat{\theta}(r\ddot{\theta} + 2\dot{r}\dot{\theta})} \qquad \text{----(10)}$$

How

acceleration in term of polar coordinate

b) Cylindrical coordinate (R, ϕ, Z) ⑳



$$\vec{r} = \hat{R} R + \hat{Z} Z \quad \text{---(1)}$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \hat{R}\dot{R} + \frac{d\hat{R}}{dt} R + \hat{Z}\dot{Z} + Z\frac{d\hat{Z}}{dt} \quad \text{--(2)}$$

$$\frac{d\hat{Z}}{dt} = 0 \quad (\hat{Z} \text{ constant})$$

From figure

$$x = R\cos\phi, \; y = R\sin\phi, \; Z = Z$$

$$\vec{R} = x\hat{i} + y\hat{j} + Z\hat{k}$$

$$\vec{R} = \hat{i}R\cos\phi + \hat{j}R\sin\phi + \hat{k}Z \quad \text{---(3)}$$

$$\frac{\partial\vec{R}}{\partial R} = \hat{i}\cos\phi + \hat{j}\sin\phi + 0 \quad \text{---(4)}$$

$$\hat{R} = \frac{\frac{\partial\vec{R}}{\partial R}}{\left|\frac{\partial\vec{R}}{\partial R}\right|} = \frac{\hat{i}\cos\phi + \hat{j}\sin\phi}{\sqrt{\cos^2\phi + \sin^2\phi}} = \hat{i}\cos\phi + \hat{j}\sin\phi \quad \text{---(5)}$$

$$\frac{\partial\hat{R}}{\partial\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi \quad \text{----(6)}$$

$$\hat{\phi} = \frac{\frac{\partial\vec{R}}{\partial\phi}}{\left|\frac{\partial\vec{R}}{\partial\phi}\right|} = \frac{-\hat{i}R\sin\phi + \hat{j}R\cos\phi}{\sqrt{R^2\sin^2\phi + R^2\cos^2\phi}} = -\hat{i}\sin\phi + \hat{j}\cos\phi \quad \text{--(7)}$$

From eq. (6) & (7) we get

$$\frac{\partial\hat{R}}{\partial\phi} = \hat{\phi}$$

by using chain rule

$$\frac{d\hat{R}}{dt} = \frac{d\hat{R}}{d\phi}\cdot\frac{d\phi}{dt} = \hat{\phi}\,\dot{\phi} \quad \text{----} (8)$$

Subs. eq. (8) in (2) we obtain

$$\boxed{\vec{V} = \hat{R}\dot{R} + \hat{\phi}R\dot{\phi} + \hat{z}\dot{z}} \quad \text{---} (9) \text{ (velocity in cylindrical coordinate)}$$

to find the acceleration

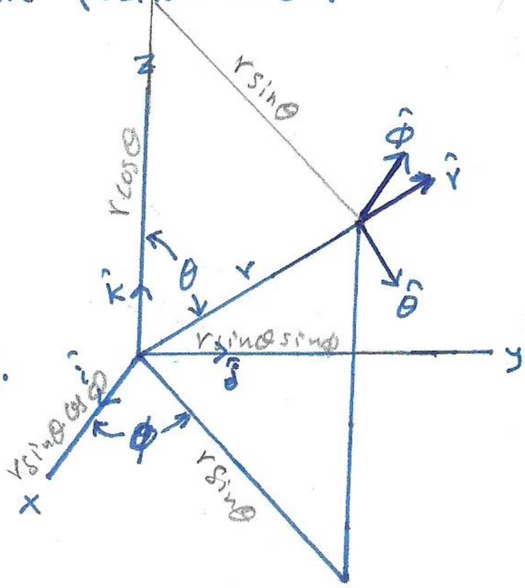
$$\vec{a} = \frac{d\vec{V}}{dt} = \hat{R}\ddot{R} + \dot{R}\frac{d\hat{R}}{dt} + \frac{d\hat{\phi}}{dt}R\dot{\phi} + \hat{\phi}\dot{R}\dot{\phi} + \hat{\phi}R\ddot{\phi} + \hat{z}\ddot{z}$$

$$\boxed{\vec{a} = \hat{R}(\ddot{R} - R\dot{\phi}^2) + \hat{\phi}(R\ddot{\phi} + 2\dot{R}\dot{\phi}) + \hat{z}\ddot{z}} \quad \text{---} (10) \text{ (acceleration in cylindrical coordinate)}$$

c) Spherical coordinate (r, θ, ϕ) : The position vector

$$\vec{r} = \hat{r}\,r \quad \text{---} (1)$$

$$\vec{V} = \hat{r}\dot{r} + r\frac{d\hat{r}}{dt} \quad \text{--} (2)$$



From eq. (8) (polar coordinate) and eq. (8) (cyl. coordinate)

eq. (2) become

$$\vec{V} = \hat{r}\dot{r} + r\left(\dot{\theta}\frac{d\hat{r}}{d\theta} + \dot{\phi}\frac{d\hat{r}}{d\phi}\right) \quad \text{---} (3)$$

From figure we have

$$x = r\sin\theta\cos\phi \;,\; y = r\sin\theta\sin\phi \;,\; z = r\cos\theta$$

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z = \hat{i}r\cos\phi\sin\theta + \hat{j}r\sin\phi\sin\theta + \hat{k}r\cos\theta$$

$$\hat{r} = \frac{\vec{r}}{|r|} = \hat{i}\cos\phi\sin\theta + \hat{j}\sin\phi\sin\theta + \hat{k}\cos\theta \quad \text{---} (4)$$

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{|\partial\vec{r}/\partial\theta|} = \hat{i}\cos\phi\cos\theta + \hat{j}\sin\phi\cos\theta - \hat{k}\sin\theta \quad \text{---} (5)$$

$$\hat{\phi} = \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|} = -\hat{i}\sin\phi + \hat{j}\cos\phi \quad \text{----} (6)$$

$$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}, \quad \frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}, \quad \frac{\partial\hat{\phi}}{\partial\theta} = 0 \quad \text{(H.w)} \quad \text{---} (7)$$

$$\frac{\partial\hat{r}}{\partial\phi} = \hat{\phi}\sin\theta, \quad \frac{\partial\hat{\theta}}{\partial\phi} = \hat{\phi}\cos\theta \quad \text{(H.w)} \quad \text{---} (8)$$

$$\frac{\partial\hat{\phi}}{\partial\phi} = (-\hat{r}\sin\theta - \hat{\theta}\cos\theta) \quad \text{---} (9)$$

$\therefore$ eq-(3) become

$$\boxed{\vec{V} = \hat{r}\dot{r} + \hat{\theta}r\dot{\theta} + \hat{\phi}r\dot{\phi}\sin\theta} \quad \text{-----} (10)$$

velocity in spherical coordinate.

The acceleration

$$\vec{a} = \frac{d\vec{V}}{dt} = \hat{r}\ddot{r} + \dot{r}\frac{d\hat{r}}{dt} + \hat{\theta}r\ddot{\theta} + \frac{d\hat{\theta}}{dt}r\dot{\theta} + \hat{\theta}\dot{r}\dot{\theta} + \hat{\phi}\frac{d}{dt}(r\dot{\phi}\sin\theta)$$