Hyperbolic functions

The hyperbolic functions $\sinh x$ and $\cosh x$ have similar names to the trigonometric functions, but they are defined in terms of the exponential function e^x. They are defined by the formula

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

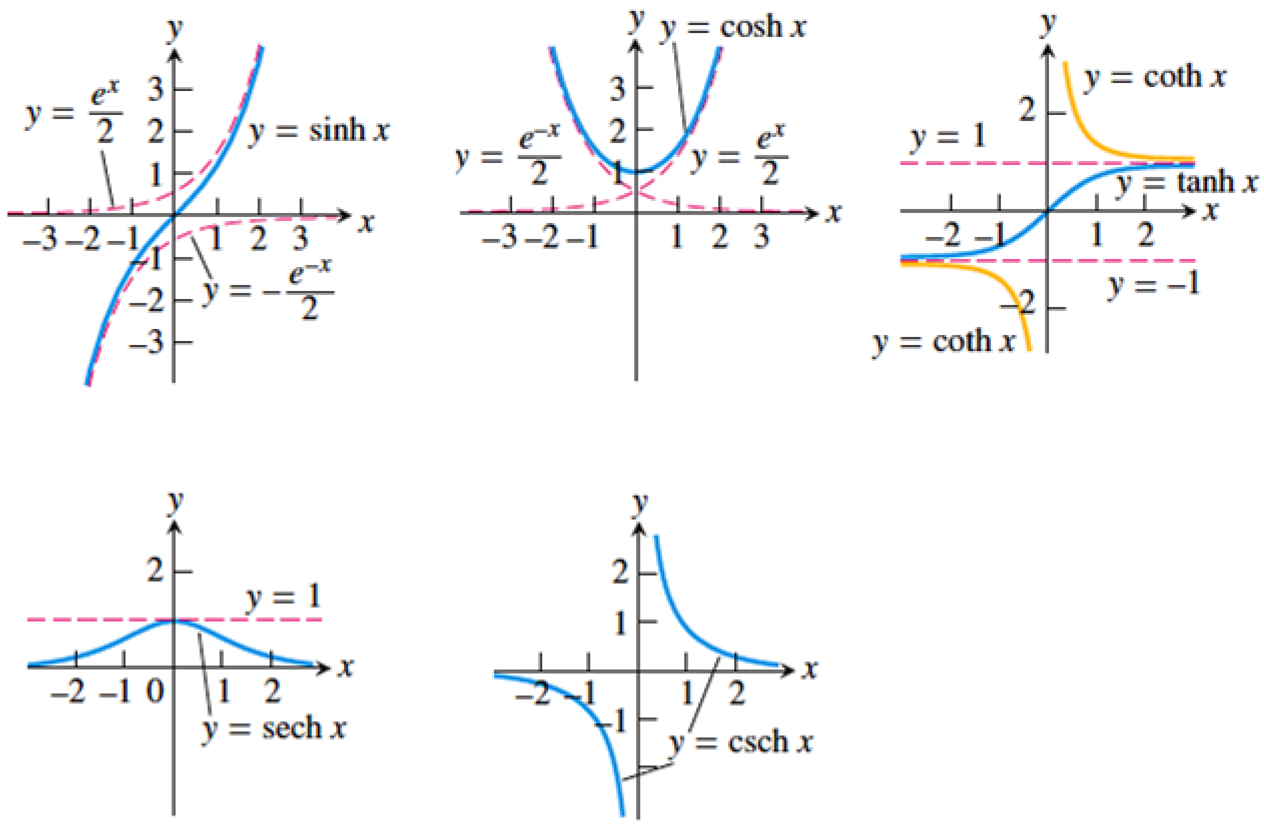
The hyperbolic function $\tanh x$ is defined by

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

We have also mentioned the reciprocal functions, and these have special names related to the names of the trigonometric reciprocal functions. They are

$$\coth x = \frac{1}{\tanh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x} \quad \text{and} \quad \operatorname{csch} x = \frac{1}{\sinh x}$$

Graphs of hyperbolic functions



Some identities for hyperbolic functions

Hyperbolic functions have identities which are similar to, but not the same as, the identities for trigonometric functions. In this section we shall prove two of these identities, and list some others.

The first identity is: $\cosh^2 x - \sinh^2 x = 1$

To prove this, we start by substituting the definitions for $\sinh x$ and $\cosh x$:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}$$

$$= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = \frac{4}{4} = 1$$

Here is another identity involving hyperbolic functions: $\sinh 2x = 2\sinh x \cosh x$

On the left-hand side we have $\sinh 2x$ so, from the definition,

$$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$$

We want to manipulate the right-hand side to achieve this. Sowe shall start by substituting the definitions of $\sinh x$ and $\cosh x$ into the right-hand side:

$$2\sinh x \cosh x = 2 \times \frac{e^x - e^{-x}}{2} \times \frac{e^x + e^{-x}}{2} = \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x$$

There are several more identities involving hyperbolic functions:

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \sinh y \cosh x$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh y \sinh x$$

If you know the trigonometric identities, you may notice that these hyperbolic identities are very similar, although sometimes plus signs have become minus signs and vice versa. In fact the hyperbolic functions are very closely related to the trigonometric functions, and $\sinh x$, $\cosh x$ are sometimes called the hyperbolic sine and hyperbolic cosine functions.

Solving equations

Suppose that $\sinh x = \frac{3}{4}$ and we wish to find the exact value of x.

$$\frac{e^x - e^{-x}}{2} = \frac{3}{4}$$

$$\{ \quad 2e^x - 2e^{-x} = 3 \quad \} \times e^x$$

$$2e^{2x} - 3e^x - 2 = 0$$

$$(2e^x + 1)(e^x - 2) = 0$$

$$e^x = 2 \quad \text{or} \quad e^x = -\frac{1}{2}$$

But e^x is always positive so $e^x = 2 \quad \Rightarrow x = \ln 2$

Example1: Solve for x: 1. $5\cosh x + 3\sinh x = 4$

$$5\left(\frac{e^x + e^{-x}}{2}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right) = 4 \quad \Rightarrow \quad \frac{5e^x + 5e^{-x} + 3e^x - 3e^{-x}}{2} = 4$$

$$\frac{8e^x + 2e^{-x}}{2} = 4 \quad \Rightarrow \quad 4e^x + e^{-x} = 4$$

$$4e^x + \frac{1}{e^x} = 4 \quad \Rightarrow \quad \frac{4e^{2x} + 1}{e^x} = 4$$

$$4e^{2x} + 1 = 4e^x \quad \Rightarrow \quad 4(e^x)^2 - 4e^x + 1 = 0$$

$$(2e^x - 1)^2 = 0 \quad \Rightarrow \quad e^x = \frac{1}{2}$$

$$x = \ln\frac{1}{2} = -\ln 2 = -0.693$$

2. $2\sinh x - 3\cosh x = -3 \quad \Rightarrow \quad \frac{2e^x - 2e^{-x} - 3e^x - 3e^{-x}}{2} = -3$

$$-e^x - 5e^{-x} = -6 \quad \Rightarrow \quad \{e^x + 5e^{-x} = 6\} \times e^x \quad \Rightarrow \quad e^{2x} - 6e^x + 5 = 0$$

$$(e^x - 1)(e^x - 5) = 0 \quad \Rightarrow \quad e^x = 1 \ \text{ or } \ e^x = 5$$

$$x = \ln 1 = 0 \ \text{ or } \ x = \ln 5 = 1.609$$

Derivatives of hyperbolic functions

If $u = u(x)$, then :

1. $\dfrac{d}{dx}(\sinh u) = \cosh u \,\dfrac{du}{dx}$

2. $\dfrac{d}{dx}(\cosh u) = \sinh u \,\dfrac{du}{dx}$

3. $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \,\dfrac{du}{dx}$

4. $\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \,\dfrac{du}{dx}$

5. $\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \,\dfrac{du}{dx}$

6. $\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \,\dfrac{du}{dx}$

Example 2: Find derivatives of the functions

1. $y = \sinh\sqrt{x^2+1} \quad \Rightarrow \quad \dfrac{dy}{dx} = \cosh\sqrt{x^2+1} \times \dfrac{2x}{2\sqrt{x^2+1}} = \dfrac{x\cosh\sqrt{x^2+1}}{\sqrt{x^2+1}}$

2. $y = \cosh(x^2-3) \quad \Rightarrow \quad \dfrac{dy}{dx} = 2x\sinh(x^2-3)$

3. $y = \tanh\sqrt{x} \quad \Rightarrow \quad \dfrac{dy}{dx} = \operatorname{sech}^2\sqrt{x} \times \dfrac{1}{2\sqrt{x}} = \dfrac{\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}}$

4. $y = e^{2x}\cosh 3x \quad \Rightarrow \quad \dfrac{dy}{dx} = 3e^{2x}\sinh 3x + 2e^{2x}\cosh 3x$

Exercises

1. Solve for x:

$a.\ \ 4\cosh x + \sinh x = 4 \qquad b.\ \ 3\sinh x - \cosh x = 1$

$c.\ \ \operatorname{sech} 2x = 0.25 \qquad d.\ \ 4\tanh x - \operatorname{sech} x = 1$

2. Find derivatives of the functions

$a.\ \ y = \cosh 2x - \sinh 3x \qquad b.\ \ y = e^{-2x}\sinh 2x$

$c.\ \ y = 3x\operatorname{sech} 2x \qquad d.\ \ y = \sqrt{x}\tanh\sqrt{x}$