8.th & 9.th weeks

Buoyancy and Stability

The same principles used to compute hydrostatic forces on surfaces can be applied to the net pressure force on a completely submerged or floating body. The results are the two laws of buoyancy discovered by Archimedes in the third century B.C.:

1. A body immersed in a fluid experiences a vertical buoyant force equal to the weight of the fluid it displaces.
2. A floating body displaces its own weight in the fluid in which it floats.

These two laws are easily derived by referring to Fig. 2.16. In Fig. 2.16*a*, the body lies between an upper curved surface 1 and a lower curved surface 2. From Eq. (2.45) for vertical force, the body experiences a net upward force

$$\begin{aligned} F_B &= F_V(2) - F_V(1) \\ &= (\text{fluid weight above } 2) - (\text{fluid weight above } 1) \\ &= \text{weight of fluid equivalent to body volume} \end{aligned} \qquad (2.48)$$

Alternatively, from Fig. 2.16*b*, we can sum the vertical forces on elemental vertical slices through the immersed body:

$$F_B = \int_{\text{body}} (p_2 - p_1)\, dA_H = -\gamma \int (z_2 - z_1)\, dA_H = (\gamma)(\text{body volume}) \qquad (2.49)$$

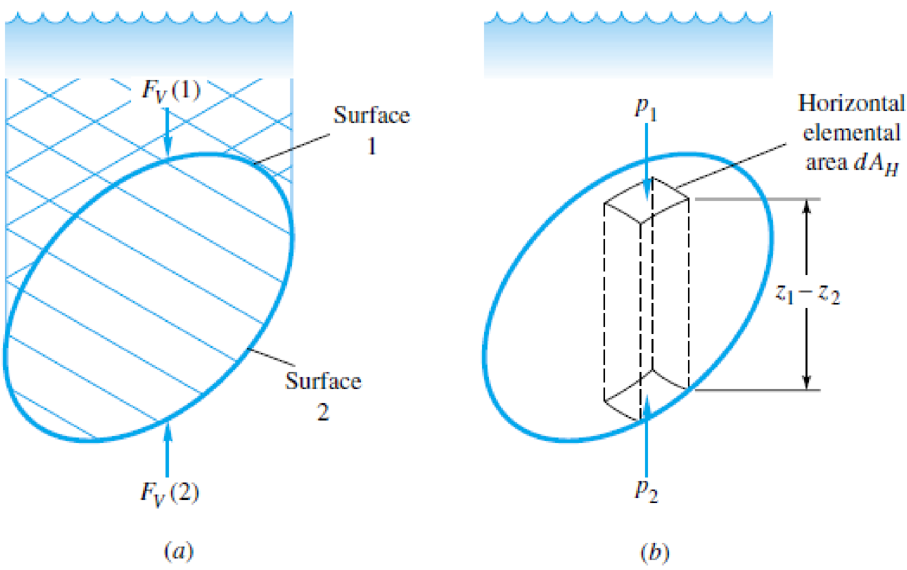


Fig. 2.16 Two different approaches to the buoyant force on an arbitrary immersed body: (*a*) forces on upper and lower curved surfaces; (*b*) summation of elemental vertical-pressure forces.

Equation (2.49) assumes that the fluid has uniform specific weight. The line of action of the buoyant force passes through the center of volume of the displaced body; i.e., its center of mass is computed as if it had uniform density. This point through which F_B acts is called the *center of buoyancy*, commonly labeled B or CB on a drawing.

Equation (2.49) can be generalized to a layered fluid (LF) by summing the weights of each layer of density ρ_i displaced by the immersed body:

$$(F_B)_{LF} = \sum \rho_i g(\text{displaced volume})_i \tag{2.50}$$

Each displaced layer would have its own center of volume, and one would have to sum moments of the incremental buoyant forces to find the center of buoyancy of the immersed body.

Floating bodies are a special case; only a portion of the body is submerged, with the remainder poking up out of the free surface. This is illustrated in Fig. 2.17, where the shaded portion is the displaced volume. Equation (2.49) is modified to apply to this smaller volume

$$F_B = (\gamma)(\text{displaced volume}) = \text{floating-body weight} \tag{2.51}$$

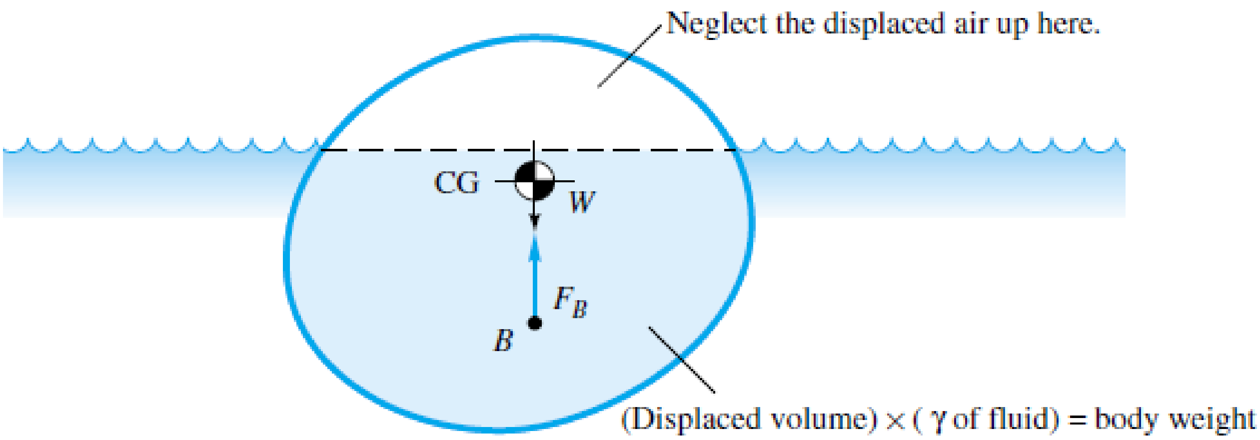


Fig. 2.17 Static equilibrium of a floating body.

EXAMPLE 2.9

A block of concrete weighs 100 lbf in air and "weighs" only 60 lbf when immersed in fresh water (62.4 lbf/ft^3). What is the average specific weight of the block?

Solution

A free-body diagram of the submerged block (see Fig. E2.9) shows a balance between the apparent weight, the buoyant force, and the actual weight

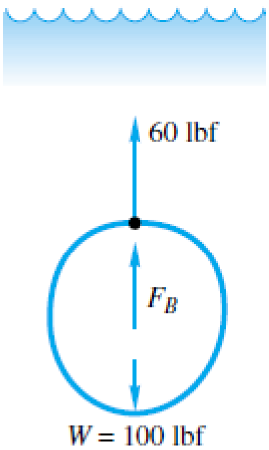
$$\sum F_z = 0 = 60 + F_B - 100$$

or

$$F_B = 40 \text{ lbf} = (62.4 \text{ lbf/ft}^3)(\text{block volume, ft}^3)$$

Solving gives the volume of the block as 40/62.4 = 0.641 ft^3. Therefore the specific weight of the block is

$$\gamma_{\text{block}} = \frac{100 \text{ lbf}}{0.641 \text{ ft}^3} = 156 \text{ lbf/ft}^3 \qquad \textit{Ans.}$$



E2.9

Stability

A floating body as in Fig. 2.17 may not approve of the position in which it is floating. If so, it will overturn at the first opportunity and is said to be statically *unstable*, like a pencil balanced upon its point. The least disturbance will cause it to seek another equilibrium position which is stable. Engineers must design to avoid floating instabil-

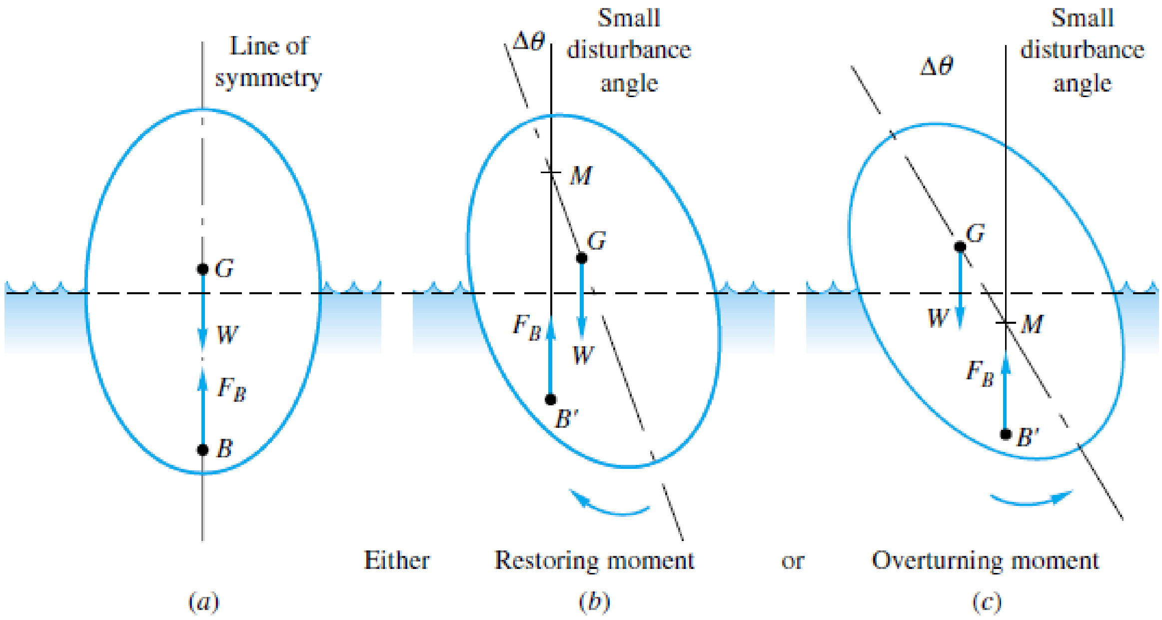


Fig. 2.18 Calculation of the metacenter M of the floating body shown in (*a*). Tilt the body a small angle $\Delta\theta$. Either (*b*) B' moves far out (point M above G denotes stability); or (*c*) B' moves slightly (point M below G denotes instability).

ity. The only way to tell for sure whether a floating position is stable is to "disturb" the body a slight amount mathematically and see whether it develops a restoring moment which will return it to its original position. If so, it is stable; if not, unstable. Such calculations for arbitrary floating bodies have been honed to a fine art by naval architects [3], but we can at least outline the basic principle of the static-stability calculation. Figure 2.18 illustrates the computation for the usual case of a symmetric floating body. The steps are as follows:

1. The basic floating position is calculated from Eq. (2.51). The body's center of mass G and center of buoyancy B are computed.
2. The body is tilted a small angle $\Delta\theta$, and a new waterline is established for the body to float at this angle. The new position B' of the center of buoyancy is calculated. A vertical line drawn upward from B' intersects the line of symmetry at a point M, called the *metacenter*, which is independent of $\Delta\theta$ for small angles.
3. If point M is above G, that is, if the *metacentric height* $\overline{MG}$ is positive, a restoring moment is present and the original position is stable. If M is below G (negative $\overline{MG}$, the body is unstable and will overturn if disturbed. Stability increases with increasing $\overline{MG}$.

Thus the metacentric height is a property of the cross section for the given weight, and its value gives an indication of the stability of the body. For a body of varying cross section and draft, such as a ship, the computation of the metacenter can be very involved.

Stability Related to Waterline Area

Naval architects [3] have developed the general stability concepts from Fig. 2.18 into a simple computation involving the area moment of inertia of the *waterline area* about the axis of tilt. The derivation assumes that the body has a smooth shape variation (no discontinuities) near the waterline and is derived from Fig. 2.19.

The y-axis of the body is assumed to be a line of symmetry. Tilting the body a small angle θ then submerges small wedge Obd and uncovers an equal wedge cOa, as shown.

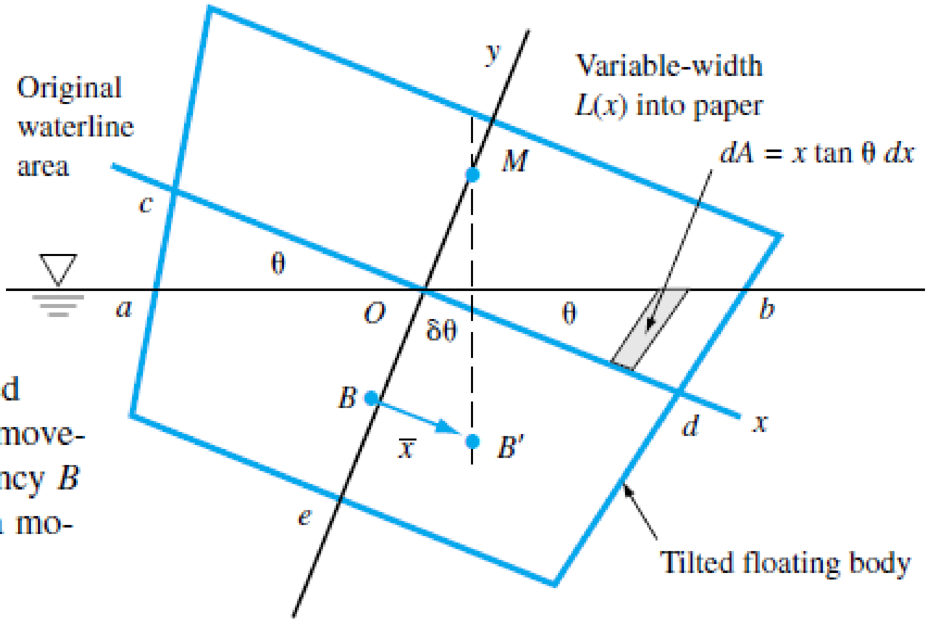


Fig. 2.19 A floating body tilted through a small angle θ. The movement $\bar{x}$ of the center of buoyancy B is related to the waterline area moment of inertia.

The new position B' of the center of buoyancy is calculated as the centroid of the submerged portion $aObde$ of the body:

$$\bar{x}\,\upsilon_{abOde} = \int_{cOdea} x\,d\upsilon + \int_{Obd} x\,d\upsilon - \int_{cOa} x\,d\upsilon = 0 + \int_{Obd} x\,(L\,dA) - \int_{cOa} x\,(L\,dA)$$

$$= 0 + \int_{Obd} x\,L\,(x \tan\theta\,dx) - \int_{cOa} xL\,(-x\tan\theta\,dx) = \tan\theta \int_{\text{waterline}} x^2\,dA_{\text{waterline}} = I_O \tan\theta$$

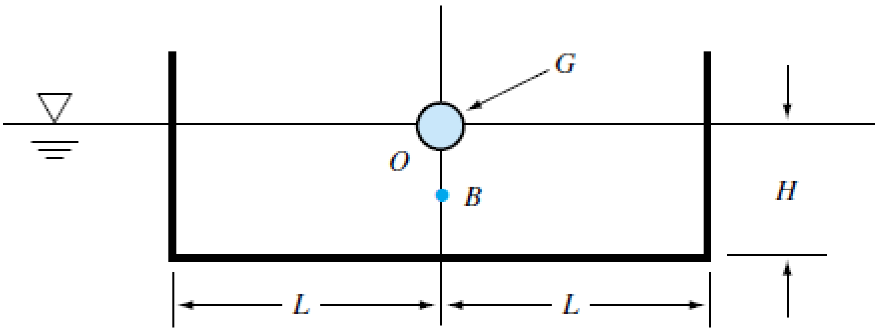
where I_O is the area moment of inertia of the *waterline footprint* of the body about its tilt axis O. The first integral vanishes because of the symmetry of the original submerged portion $cOdea$. The remaining two "wedge" integrals combine into I_O when we notice that $L\,dx$ equals an element of *waterline area*. Thus we determine the desired distance from M to B:

$$\frac{\bar{x}}{\theta} = \overline{MB} = \frac{I_O}{\upsilon_{\text{submerged}}} = \overline{MG} + \overline{GB} \qquad \text{or} \qquad \overline{MG} = \frac{I_O}{\upsilon_{\text{sub}}} - \overline{GB} \qquad (2.52)$$

The engineer would determine the distance from G to B from the basic shape and design of the floating body and then make the calculation of I_O and the submerged volume υ_{sub}. If the metacentric height MG is positive, the body is stable for small disturbances. Note that if $\overline{GB}$ is negative, that is, B is *above* G, the body is always stable.

EXAMPLE 2.10

A barge has a uniform rectangular cross section of width $2L$ and vertical draft of height H, as in Fig. E2.10. Determine (*a*) the metacentric height for a small tilt angle and (*b*) the range of ratio L/H for which the barge is statically stable if G is exactly at the waterline as shown.



E2.10

Solution

If the barge has length b into the paper, the waterline area, relative to tilt axis O, has a base b and a height $2L$; therefore, $I_O = b(2L)^3/12$. Meanwhile, $\upsilon_{sub} = 2LbH$. Equation (2.52) predicts

$$\overline{MG} = \frac{I_O}{\upsilon_{sub}} - \overline{GB} = \frac{8bL^3/12}{2LbH} - \frac{H}{2} = \frac{L^2}{3H} - \frac{H}{2} \qquad \textit{Ans. (a)}$$

The barge can thus be stable only if

$$L^2 > 3H^2/2 \qquad \text{or} \qquad 2L > 2.45H \qquad \textit{Ans. (b)}$$

The wider the barge relative to its draft, the more stable it is. Lowering G would help also.