Chapter One

Mathematical Logic **المنطق الرياضياتي**

علم المنطق هو أحد فروع علم الرياضيات الصرفة وهو علم حديث نسبيا ً، وقد أخذت أهميته تتزايد يوما ً بعد يوم.
يفهم من إسم هذا العلم أنه يشارك اللغات في وظائفها ومدلولاتها وتعبيراتها، فعلم المنطق يرتكز على مبادىء واضحة متفق عليها عالميا ً، وله رموز خاصة به، ومن الجدير بالذكر أن كل علم أو بالاحرى كل فرع من فروع المعرفة له ألفاظ ومصطلحات خاصة به، إلا ان هذه الالفاظ ربما لا تستخدم في حديثنا اليومي، وقد يستخدم بعضها بمعنى مقارب لما تعنيه في حديثنا اليومي، وفي حالات يختلف معناها تماما ً عن مقصودنا وذلك ربما يرجع الى عدم دقة التعبير عندنا وليس معناه قصورا ً في اللغة المستخدمة في التعبير.
ولما كان هذا الابهام غير مرغوب فيه وبخاصة في الرياضيات، فلم يترك الامر للإجتهاد في المنطق الرياضياتي، بل اتفق على رموز و أدوات لربط الجمل واعطيت معاني محددة تماما ً لاتقبل اللبس والغموض وهذا يقودنا الى القول بان المنطق الرياضياتي لغة علمية متفق عليها بين الرياضياتيين، ولاغنى للرياضيات عن المنطق. فالرياضيات تحتاج الى تفكير منطقي ولا يكون برهان صيغة او مبرهنة رياضياتية مثلا ً سهلا ً ومقبولا ً ما لم يستند في خطواته على سلسلة من الافكار مرتبط بعضها ببعض.

<u>العبارات (التقارير): Statements</u>

We know that the sentences in the Arabic language, including what are an actual and what are nominal ones and ones What Astvhammep or request ... etc. In mathematical logic divided into two sentences:

- It is news sentences that carry a newsflash.
- sentences is not news (construction) that does not carry a particular news.

<u>Definition (1):</u>
Each sentence that carries news that can be judged as either (True) or (False), but does not become true or false as the same time and can be called statemet (report).

<u>Definition (2):</u>
Each correct news sentence is called correct statement and each false news sentence is called false statement.

<u>Example (1):</u>
(1) the sun rises from the east. news sentence (correct statement).
(2) Baghdad is the capital of the Republic of Iraq. news sentence (correct statement).
(3) $17 < 14$. news sentence (false statement).
(4) How beautiful this grove. sentence not news (Wonder).
(5) Nawaf be careful on doing good. sentence not news (call).
(6) $3 + x = 7$, where x is an integer. news sentence.

(we can't judge it as true or false unless you know the value of the variable x). Statements from this type called (Propositional Functions).

Negation of Statements نفي العبارة (التقرير)

If we wanted to negation the statement "it's raining today", we say "it doesn't rain today" If the statement to negation is true the negation become false statement and vice versa.

Example (2):

(1) $2+3=8$: false statement, negation is $2+3\neq8$, true statement.

(2) Baghdad capital of Iraq: true statement, negation, Baghdad not capital of Iraq. false statement.

Often, we symbolized the statement as the letter of the alphabet to ease In example (2) If we symbol for the statement contained in paragraph (1) under the symbol P, we symbolized to negation this statement symbol ~P (read negation P) and that's where the statement P and ~P impossible to be true or false at the same time, we had made the letter T symbolizes the word (True) and letter F symbolizes the word (False), for we can generate a table called the right table (truth table) describes the P and ~P together as shown in the following table:

P	~ P
T	F
F	T

Notes:

(1) known as T and F true value (or truth) and replace all of them at times to 1 and 0 respectively.

(2) Note that whatever the statement P, they either have to take the value T or F value, the value of the true statement ~ P must be opposite to the value of P is true as we pointed this previously.

Definition (3):

Every statement carrying one news called a **simple statement** (primary), but if the statement carried the two or more than news called the **compound statement**.

Example (3):

(1) water freezes at zero degrees and boils at 100 degrees. (compound statement).

(2) Fawaz studies mathematics or geography. (compound statement).

(3) If the $3 + 1 = 4$, the $6 + 7 = 13$. (compound statement)

(4) abc equilateral triangle if and only if it was Equiangular. (compound statement).

Connectives :أدوات الربط

1. AND

It is symbolized by the mathematical symbol "$\wedge$". In other words, if each of the A, B, the statement "A and B" is the compound statement represented by the symbol "$A \wedge B$". (Or knew was agreed) to be the right values for this compound statement, as in Table 1.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Table (1)

From Table 1 we note that the statement $A \wedge B$ true in only one case, which is when they are two symbols A and B are true together, and false otherwise.

Example (4):

Assuming that A, B, C, D are respectively the following statements:
$2 + 2 = 4$, the moon orbits around Mars, passing the Euphrates River in Iraq, $3 = 0$.
We find that A, C true statements, while B, D false statements. Thus by reference to the Table (1) conclude that:
$A \wedge C$ is a true statement, but the $A \wedge B$, $A \wedge D$, $D \wedge B$, $B \wedge D$, $(A \wedge D) \wedge C$ are all false statements.

2.OR

It is symbolized by the mathematical symbol "$\vee$". In other words, if each of the A, B, the statement "A or B" is the compound statement represented by the symbol "$A \vee B$". (Or knew was agreed) to be the right values for this compound statement, as in Table 2.

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Table (2)

Example (5):

(1) $5 + 1 = 6$ or $3 \times 4 = 12$. correct statement.
(2) 9 an even number or 9 an odd number . correct statement.

(3)Riyadh, the capital of Syria or Delhi, the capital of Algeria. false statement.

3. If…then…

It is symbolized by the mathematical symbol "$\longrightarrow$". In other words, if each of the A, B, the statement. The conditional sentence "if A, then B" is the compound statement represented by the symbol "A $\longrightarrow$ B". (Or knew was agreed) to be the right values for this compound statement, as in Table 3.

A	B	A $\longrightarrow$ B
T	T	T
T	F	F
F	T	T
F	F	T

Table (3)

See from table 3 that the compound statement A $\longrightarrow$ B be false in only one case when the A true and B false.

Example (6):

(1) $5 + 7 = 12 \longrightarrow 2 + 6 = 8$ T
(2) $5 + 7 = 11 \longrightarrow 2 + 6 = 8$ T
(3) $5 + 7 = 11 \longrightarrow 2 + 6 \neq 8$ T
(4) $5 + 7 = 12 \longrightarrow 2 + 6 = 7$ F

4. If and only if

It is symbolized by the mathematical symbol "$\leftrightarrow$". In other words, if each of the A, B, the statement. "A if and only if B" is the compound statement represented by the symbol "A $\leftrightarrow$B". Thus the statement A $\leftrightarrow$B can be expressed as:

$$(A \longrightarrow B) \wedge (B \longrightarrow A)$$

For that we set true table in terms of tables (1) and (3) as follows:

A	B	A $\longrightarrow$ B	B $\longrightarrow$ A	(A $\longrightarrow$ B) $\wedge$ (B $\longrightarrow$ A)	A $\longleftrightarrow$ B
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Table (4)

Notes from the table (4) that the phrase $A \leftrightarrow B$ are correct when they are two statements A and B true together or false together.

Example (7):

(1) $5 + 3 = 8 \leftrightarrow 5 \times 3 = 15$. T

(2) Iraq is located in Europe, $\leftrightarrow 5 + 3 = 8$. F

(3) $5\times3 = 15 \leftrightarrow$ Fatima man's name. F

(4) Sanaa, the capital of Russia's $\leftrightarrow$ sugar tastes bitter. T

Definition (4):

Said of the two statements A and B, they are equivalent logically, or become are equivalent if each of them have the same table or the right values, and so represented by the symbol $A \equiv B$, (reading A equivalent to B).

Example (8):

(1) $(A \longrightarrow B) \leftrightarrow (B \longrightarrow A) \equiv$ (Table 4).

(2) $A \equiv A \wedge A \equiv A \vee A \equiv \sim (\sim A)$ as in the following table.

A	**A**	**~A**	**$A \wedge A$**	**$A \vee A$**	**~(~A)**
T	T	F	T	T	T
F	F	T	F	F	F

Theorem (1): De Morgan's Laws

Whatever the two statements, the A and B:

(A) $\sim (A \wedge B) \equiv (\sim A) \vee (\sim B)$

(B) $\sim (A \vee B) \equiv (\sim A) \wedge (\sim B)$

proof:

(A) Prove If the true values for the statement $\sim (A \wedge B)$ are the same true values to the statements $(\sim A) \vee (\sim B)$ ،as accordance with the definition (4) so we construct (Table 5).

A	**B**	**~A**	**~B**	**$A \wedge B$**	**$\sim(A \wedge B)$**	**$(\sim A) \vee (\sim B)$**
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Table (5)

From columns sixth and seventh we see the equally of the right values and thus was required.
(B) to prove is required in the same way (a) **(leave to the student).**

<u>Note:</u>
It can prove the validity of paragraph (B) from the theorem (1) in another way as follows:
Negation the right end of the relationship (B), we find that:

$\sim[(\sim A) \wedge (\sim B)] \equiv \sim(\sim A) \vee \sim(\sim B)$ According to paragraph (A) of the theorem.
$\equiv A \vee B$ paragraph (2) Example (8)
$\sim[(\sim A) \wedge (\sim B)] \equiv A \vee B$ $(*)$
Negation the relationship $(*)$ we get the required proved which:
$\sim(\sim[(\sim A) \wedge (\sim B)] \equiv \sim(A \vee B)$
Meaning that:
$(\sim A) \wedge \sim B) \equiv \sim(A \vee B)$
<u>Theorem (2):</u>

If A, any two statements, the:

$$A \longrightarrow B \equiv \sim(A \wedge \sim B)$$

<u>proof:</u>

According definition (4) is enough to create table (6), in which we see that the fifth and sixth columns are equal in right values so desired proved.

A	**B**	**$\sim B$**	**$A \wedge \sim B$**	**$A \longrightarrow B$**	**$\sim(A \wedge \sim B)$**
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Table (6)

<u>Definition (5):</u>
It said about the compound statement is is a logically correct if all correct values true. And said is logically false if all correct values false.
<u>Example (9):</u>
(1) statement $A \vee \sim A$. logically true.
(2) statement $A \wedge \sim A$. logically false.
The following table proved paragraphs (1) and (2) according to definition (5).

A	**$\sim A$**	**$A \vee \sim A$**	**$A \wedge \sim A$**

T	F	T	F
F	T	T	F

Note:

Statement may not be logically correct or logically false, as in the phrases $A \longrightarrow B$, $B \longleftrightarrow A$, for example.

Definition (6):

We said that statement A lead to statement B, and represented that by symbol $A \Rightarrow B$, if the statement $A \longrightarrow B$ logically correct. As sometimes we say that A is introduction and B is the result.

Example (10):

(1) Whatever the statement A, the $A \Rightarrow A \vee \sim A$, because $A \longrightarrow A \vee \sim A$ is a logically correct as shown in example (9), paragraph (1).

(2) Whatever the statements A, B, the $A \Rightarrow A \vee B$ because $A \longrightarrow A \vee B$ is a logically correct. (Check up leave to the student).

(3) Whatever the statements A, B, the $A \wedge B \Rightarrow A \vee B$ because it is easy to verify that the compound statement $A \wedge B \longrightarrow A \vee B$ is a logically correct statement (check it out).

Notes:

(1) If $A \Rightarrow B$ A , the correct table for the conpound statement, $A \longrightarrow B$ note that:

(A) Whenever the statement A correct the statement B correct too.

(B) Whenever the statement B the false statement A false too.

In other words if the introduction A correct, the result B is also correct, and if the result B is false, the introduction A false too.

(2) If $A \Rightarrow B$ we express it by saying that, if the statement A correct, it is enough to be the statement B correct too).

(3) when A it does not lead to the B, so we represented by the symbol $A \nRightarrow B$.

(4) $A \Rightarrow B$ does not have correct table, because we did not consider the symbol "$\Rightarrow$" link between the two statements A, B.

Definition (7):

A statement we say that lead to statement B, if B statement leading to the statement A, and symbolized $A \Leftrightarrow B$, if statement $B \longleftrightarrow A$ logically correct.

The symbol "$\Leftrightarrow$" Not link between the two statements A, B. Therefore, $A \Leftrightarrow B$ is not correct table. It should also be noted that we sometimes express the symbol "$\Leftrightarrow$"by saying "the necessary and sufficient condition". It also means equivalent to the word. And sometimes it can be used instead of the symbol "$\equiv$" as illustrated by the following example.

Example (11):

Whatever the statements A, B is: $\sim(A \longrightarrow B) \Leftrightarrow A \wedge \sim B$

Solution: create table for the statement $\sim(A \longrightarrow B) \Leftrightarrow A \wedge \sim B$ as follows:

A	B	~ B	$A \longrightarrow B$	$A \wedge \sim B$	$\sim(A \longrightarrow B)$	$\sim(A \longrightarrow B) \leftrightarrow A \wedge \sim B$
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

Note from the table the statement $\sim(A \longrightarrow B) \leftrightarrow A \wedge \sim B$ logically correct as shown in the seventh column. As the right values in columns fifth and sixth in the table equal, which is consistent with the definition of parity that any $\sim(A \longrightarrow B) \leftrightarrow A \wedge \sim B$ we would consider that the symbols $\Leftrightarrow$ ، $\equiv$ have the same meaning.

Here it should be noted that if it is not considered $A \Leftrightarrow B$ accrued, we symbolized so the symbol $A \not\Leftrightarrow B$.

Example (12):

Verification the relationship between $x = 3 \Rightarrow x^2 = 9$ or not, taking advantage of the comments received after the example (10).

Solution:

(A) The first method: our knowledge of mathematical, we know that when the statement would be $x = 3$ correct it necessarily lead to that statement $x^2 = 9$ correct. Since it can not be $x = 3$ while the $x^2 \neq 9$ Thus, $x = 3 \Rightarrow x^2 = 9$ verification.

(**B) The second method:** From our information also, we know that when the statement would be $x^2 = 9$ false, i.e. when $x^2 \neq 9$ the statement $x = 3$ be false, which means that $x \neq 3$ so the $x = 3 \Rightarrow x^2 = 9$ verification.

Before produced the following theorem, which includes most of the properties of the two links "$\wedge$" and "$\vee$" We would like to note the following note.

Note:

If we have one statement, the number of possible correct values two, and if we have two different statement, the number of possible correct values of four, and if we have three different statements, the number of possible correct values eight, this can be the proof that if we have n different statements, the number of possible correct values equal to 2^n that

$B(n) = 2^n; \quad n \in \mathbb{N}$

Theorem (3):

Whenever the statements A ،B ،C , thus,

(1) $A \wedge A \equiv A$ as well as $A \vee A \equiv A$.خاصية اللانمو

(2) $A \wedge B \equiv B \wedge A$ as well as $A \vee B \equiv B \vee A$ substitution property.

(3) $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$ as well as $(A \vee B) \vee C \equiv A \vee (B \vee C)$ property (respectively) the merger.

(4) $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ distribution of property.

$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ distribution of property.

proof:

We prove that "$\wedge$" link distributed on the link "$\vee$" (leaving the rest of the proofs on the health properties mentioned in the theorem on the student) for that creating the following table and conclude that it's health is required, as shown in the two columns seventh and eighth.

A	**B**	**C**	**$A \wedge B$**	**$A \wedge C$**	**$B \vee C$**	**$A \wedge (B \vee C)$**	**$(A \wedge B) \vee (A \wedge C)$**
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	T	F	F
F	T	F	F	F	T	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

exercises

(1) If the A representation of the statement "the rain came down: and B representation for the statement " Akhaddrt ground, "write the verbal translator for each of the following:

(a) $A \wedge B$ (b) $A \vee B$ (c) $\sim A \wedge B$ (d) $A \longrightarrow B$

(e) $B \longleftrightarrow A$ (f) $\sim A \longrightarrow B$ (g) $\sim A \longrightarrow \sim B$

(2) If p and q are two statement proving that:

$p \longrightarrow q \equiv (\sim p) \vee q \equiv \sim(p \wedge \sim q) \equiv \sim q \longrightarrow \sim p$

(3) proved that the following statements correct logically:

(a) $A \longrightarrow A \vee A$ (b) $A \longrightarrow A \vee B$ (c) $A \wedge B \longrightarrow A$

(d) $A \wedge B \longrightarrow B \wedge A$ (e) $[(A \longrightarrow B) \wedge (B \longrightarrow C)] \longrightarrow (A \longrightarrow C)$

(4) proved that the following statement is not correct logical lyor false logically
$A \longrightarrow A \wedge B$

(5) If the D, E, K three statements imposed , proving that:
$D \vee (E \vee K) \equiv (D \vee E) \vee (D \vee K)$