ARGUMENTS

An *argument* is an assertion that a given set of propositions $P_1, P_2, \ldots, P_n$, called *premises*, yields (has a consequence) another proposition Q, called the *conclusion*. Such an argument is denoted by

$$P_1,\ P_2,\ \ldots,\ P_n \vdash Q$$

The notion of a "logical argument" or "valid argument" is formalized as follows:

Definition 4.4: An argument $P_1,\ P_2,\ \ldots,\ P_n \vdash Q$ is said to be *valid* if Q is true whenever all the premises $P_1, P_2, \ldots, P_n$ are true.

An argument which is not valid is called *fallacy*.

EXAMPLE

(a) The following argument is valid:

$$p,\ p \rightarrow q \vdash q \quad (\textit{Law of Detachment})$$

The proof of this rule follows from the truth table Specifically, p and $p \rightarrow q$ are true simultaneously only in Case (row) 1, and in this case q is true.

(b) The following argument is a fallacy:

$$p \rightarrow q,\ q \vdash p$$

For $p \rightarrow q$ and q are both true in Case (row) 3 in the truth table in , but in this case p is false.

Now the propositions $P_1, P_2, \ldots, P_n$ are true simultaneously if and only if the proposition $P_1 \wedge P_2 \wedge \ldots P_n$ is true. Thus the argument $P_1, P_2, \ldots, P_n \vdash Q$ is valid if and only if Q is true whenever $P_1 \wedge P_2 \wedge \ldots \wedge P_n$ is true or, equivalently, if the proposition $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow Q$ is a tautology. We state this result formally.

Theorem : The argument $P_1,\ P_2,\ \ldots,\ P_n \vdash Q$ is valid if and only if the proposition $(P_1 \wedge P_2 \ldots \wedge P_n) \rightarrow Q$ is a tautology.

We apply this theorem in the next example.

EXAMPLE A fundamental principle of logical reasoning states:

"If p implies q and q implies r, then p implies r"

p	q	r	$[(p$	$\rightarrow$	$q)$	$\wedge$	$(q$	$\rightarrow$	$r)]$	$\rightarrow$	$(p$	$\rightarrow$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	F	F	T	T	F	F
T	F	T	T	F	F	F	F	T	T	T	T	T	T
T	F	F	T	F	F	F	F	T	F	T	T	F	F
F	T	T	F	T	T	T	T	T	T	T	F	T	T
F	T	F	F	T	T	F	T	F	F	T	F	T	F
F	F	T	F	T	F	T	F	T	T	T	F	T	T
F	F	F	F	T	F	T	F	T	F	T	F	T	F
Step			1	2	1	3	1	2	1	4	1	2	1

That is, the following argument is valid:

$$p \rightarrow q,\ q \rightarrow r \vdash p \rightarrow r \quad \textit{(Law of Syllogism)}$$

This fact is verified by the truth table .   which shows that the following proposition is a tautology:

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

Example:

Let Q_1 = "John graduates"

Q_2 = "Mary graduates"

Q_3 = "John gets a job"

Q_4 = "Mary gets a job"

Q_5 = "Mary earns money"

(i) Consider the following argument:

"If John graduates then he gets a job".

"John graduates".

"Therefore John gets a job".

To see the "form" of this argument we symbolize it as $Q_1 \rightarrow Q_3, Q_1 \vdash Q_3$.

Now we check that $((Q_1 \rightarrow Q_3) \wedge Q_1) \rightarrow Q_3$ is a tautology:

Q_1	Q_3	$Q_1 \rightarrow Q_3$	$(Q_1 \rightarrow Q_3) \wedge Q_1$	$((Q_1 \rightarrow Q_3) \wedge Q_1) \rightarrow Q_3$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

This tautology shows that the argument is valid.

Example Consider the following argument:

"If Mary graduates then she gets a job".
"Mary does not get a job".
"Therefore, Mary does not graduate".

Symbolized, this becomes: $Q_2 \to Q_4, (\neg Q_4) \vdash (\neg Q_2)$.
Now check that $((Q_2 \to Q_4) \wedge (\neg Q_4)) \vdash (\neg Q_2)$ is a tautology.

Q_2	Q_4	$Q_2 \to Q_4$	$\neg Q_4$	$(Q_2 \to Q_4) \wedge (\neg Q_4)(\equiv A)$	$\neg Q_2$	$A \to (\neg Q_2)$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Example Consider the following argument:

"If Mary graduates then she gets a job".
"If Mary gets a job then she earns money".
"Therefore if Mary graduates then she earns money".

Symbolized, this becomes $Q_2 \to Q_4, Q_4 \to Q_5 \vdash (Q_2 \to Q_5)$.
*Now check that $((Q_2 \to Q_4) \wedge (Q_4 \to Q_5)) \to (Q_2 \to Q_5)$ is a tautology.

Q_2	Q_4	Q_5	$Q_2 \to Q_4$ A	$Q_4 \to Q_5$ B	$A \wedge B$	$Q_2 \to Q_5$ C	$(A \wedge B) \to C$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

We can sum up the above by saying the following are all valid:
(i) $p \to q, p \vdash q$,
(ii) $p \to q, \neg q \vdash \neg p$,
(iii) $p \vee q, \neg q \vdash p$,
(iv) $p \to q, q \to r \vdash p \to r$.

Example Show that $p \to q, q \vee p \vdash (\neg q) \vee (\neg p)$ is valid.

p	q	$p \to q$	$q \vee p$	$((p \to q) \wedge (q \vee p))$ P	$((\neg q) \vee (\neg p))$ C	$P \to C$	
T	T	T	T	T	F	F	$\leftarrow$
T	F	F	T	F	T	T	
F	T	T	T	T	T	T	
F	F	T	F	F	T	T	

Homework:

Is $p \to q, q \vee p \vdash (\neg q) \vee (\neg p)$ valid?

Is $p \to (s \to (\neg r)), p \to r, p \vdash \neg s$ valid?

Is $(p \vee q) \to s, q \to s \vdash s$ valid?