



**Stage:** First.

**Topic:** Logic Design I

**Lecture2:** BOOLEAN ALGEBRA AND LOGIC GATES.

**General Objective:**

The students will be able to deal with Boolean Algebra and logic gates.

**Procedural Objective:**

- 1- The students can work with Boolean algebra and making use of its rules.
- 2- The students will be able to simplify any given expression.
- 3- The students can deal with logic gates and making use of it to build any desired circuit.

**Teaching aids:** Lecture (presented by MS-power point program on the laptop), discussion between lecturer and students, LCD monitor, whiteboard.

# **BOOLEAN ALGEBRA AND LOGIC GATES**

## **2.1 Introduction**

Binary logic deals with variables that have two discrete values: 1 for *TRUE* and 0 for *FALSE*. A simple switching circuit containing active elements such as a diode and transistor can demonstrate the binary logic, which can either be ON (switch closed) or OFF (switch open). Electrical signals such as voltage and current exist in the digital system in either one of the two recognized values, except during transition.

The switching functions can be expressed with Boolean equations. Complex Boolean equations can be simplified by a new kind of algebra, which is popularly called Switching Algebra or Boolean Algebra, invented by the mathematician George Boole in 1854. Boolean Algebra deals with the rules by which logical operations are carried out.

## **2.2 BASIC DEFINITIONS**

Boolean algebra, like any other deductive mathematical system, may be defined with a set of elements, a set of operators, and a number of assumptions and postulates. A set of elements means any collection of objects having common properties. If  $S$  denotes a set, and  $X$  and  $Y$  are certain objects, then  $X \in S$  denotes  $X$  is an object of set  $S$ , whereas  $Y \notin S$  denotes  $Y$  is not the object of set  $S$ . A binary operator defined on a set  $S$  of elements is a rule that assigns to each pair of elements from  $S$  a unique element from  $S$ . As an example, consider this relation  $X*Y = Z$ . This implies that  $*$  is a binary operator if it specifies a rule for finding  $Z$  from the objects  $(X, Y)$  and also if all  $X, Y$ , and  $Z$  are of the same set  $S$ . On the other hand,  $*$  can not be binary operator if  $X$  and  $Y$  are of set  $S$  and  $Z$  is not from the same set  $S$ .

The postulates of a mathematical system are based on the basic assumptions, which make possible to deduce the rules, theorems, and properties of the system. Various algebraic structures are formulated on the basis of the most common postulates, which are described as follows:

**1. Closer:** A set is closed with respect to a binary operator if, for every pair of elements of  $S$ , the binary operator specifies a rule for obtaining a unique element of  $S$ . For example, the set of natural numbers  $N = \{1, 2, 3, 4, \dots\}$  is said to be closed with respect to the binary operator plus  $(+)$  by the rules of arithmetic addition, since for any  $X, Y \in N$  we obtain a unique element  $Z \in N$  by the operation  $X + Y = Z$ . However, note that the set of natural numbers is not closed with respect to the binary operator minus  $(-)$  by the rules of arithmetic subtraction because for  $1 - 2 = -1$ , where  $-1$  is not of the set of natural numbers.

**2. Associative Law:** A binary operator  $*$  on a set  $S$  is said to be associated whenever  $(A*B)*C = A*(B*C)$  for all  $A, B, C \in S$ .

**3. Commutative Law:** A binary operator  $*$  on a set  $S$  is said to be commutative Whenever  $A*B = B*A$  for all  $A, B \in S$ .

**4. Identity Element:** A set  $S$  is to have an identity element with respect to a binary operation  $*$  on  $S$ , if there exists an element  $E \in S$  with the property  $E * A = A * E = A$ .

**Example:** The element 0 is an identity element with respect to the binary operator  $+$  on the set of integers  $I = \{ \dots -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots \}$  as  $A + 0 = 0 + A = A$ .

Similarly, the element 1 is the identity element with respect to the binary operator  $\times$  as  $A \times 1 = 1 \times A = A$ .

**5. Inverse:** If a set  $S$  has the identity element  $E$  with respect to a binary operator  $*$ , there exists an element  $B \in S$ , which is called the inverse, for every  $A \in S$ , such that  $A * B = E$ .

**Example:** In the set of integers  $I$  with  $E = 0$ , the inverse of an element  $A$  is  $(-A)$  since  $A + (-A) = 0$ .

**6. Distributive Law:** If  $*$  and  $(.)$  are two binary operators on a set  $S$ ,  $*$  is said to be distributive over  $(.)$ , whenever

$$A * (B.C) = (A * B).(A * C).$$

If summarized, for the field of real numbers, the operators and postulates have the following meanings:

The binary operator  $+$  defines addition.

The additive identity is 0.

The additive inverse defines subtraction.

The binary operator  $(.)$  defines multiplication.

The multiplication identity is 1.

The multiplication inverse of  $A$  is  $1/A$ , defines division *i.e.*,  $A \cdot 1/A = 1$ .

The only distributive law applicable is that of  $(.)$  over  $+$

$$A \cdot (B + C) = (A \cdot B) + (A \cdot C)$$

### **2.3 DEFINITION OF BOOLEAN ALGEBRA**

In 1854 George Boole introduced a systematic approach of logic and developed an algebraic system to treat the logic functions, which is now called Boolean algebra. In 1938 C.E. Shannon developed a two-valued Boolean algebra called Switching algebra, and demonstrated that the properties of two-valued or bistable electrical switching circuits can be represented by this algebra. The postulates formulated by E.V. Huntington in 1904 are employed for the formal definition of Boolean algebra. However, Huntington postulates are not unique for defining Boolean algebra and other postulates are also used. The following Huntington postulates are satisfied for the definition of Boolean algebra on a set of elements  $S$  together with two binary operators  $(+)$  and  $(.)$ .

1. (a) Closer with respect to the operator  $(+)$ .

(b) Closer with respect to the operator  $(.)$ .

2. (a) An identity element with respect to  $+$  is designated by 0 *i.e.*,  $A + 0 = 0 + A = A$ .

- (b) An identity element with respect to  $\cdot$  is designated by 1 *i.e.*,  $A \cdot 1 = 1 \cdot A = A$ .
3. (a) Commutative with respect to  $(+)$ , *i.e.*,  $A + B = B + A$ .  
 (b) Commutative with respect to  $(\cdot)$ , *i.e.*,  $A \cdot B = B \cdot A$ .
4. (a)  $(\cdot)$  is distributive over  $(+)$ , *i.e.*,  $A \cdot (B+C) = (A \cdot B) + (A \cdot C)$ .  
 (b)  $(+)$  is distributive over  $(\cdot)$ , *i.e.*,  $A + (B \cdot C) = (A + B) \cdot (A + C)$ .
5. For every element  $A \in S$ , there exists an element  $A' \in S$  (called the complement of A) such that  $A + A' = 1$  and  $A \cdot A' = 0$ .
6. There exists at least two elements  $A, B \in S$ , such that A is not equal to B.

Comparing Boolean algebra with arithmetic and ordinary algebra (the field of real numbers), the following differences are observed:

- Huntington postulates do not include the associate law. However, Boolean algebra follows the law and can be derived from the other postulates for both operations.
- The distributive law of  $(+)$  over  $(\cdot)$  *i.e.*,  $A + (B \cdot C) = (A+B) \cdot (A+C)$  is valid for Boolean algebra, but not for ordinary algebra.
- Boolean algebra does not have additive or multiplicative inverses, so there are no subtraction or division operations.
- Postulate 5 defines an operator called Complement, which is not available in ordinary algebra.
- Ordinary algebra deals with real numbers, which consist of an infinite set of elements. Boolean algebra deals with the as yet undefined set of elements S, but in the two valued Boolean algebra, the set S consists of only two elements: 0 and 1.

Boolean algebra is very much similar to ordinary algebra in some respects. The symbols  $(+)$  and  $(\cdot)$  are chosen intentionally to facilitate Boolean algebraic manipulations by persons already familiar to ordinary algebra. Although one can use some knowledge from ordinary algebra to deal with Boolean algebra, beginners must be careful not to substitute the rules of ordinary algebra where they are not applicable.

## 2.4 TWO-VALUED BOOLEAN ALGEBRA

Two-valued Boolean algebra is defined on a set of only two elements,  $S = \{0,1\}$ , with rules for two binary operators  $(+)$  and  $(\cdot)$  and inversion or complement as shown in the following operator tables at Figures 2.1, 2.2, and 2.3 respectively.

A	B	A + B
0	0	0
0	1	1
1	0	1
1	1	1

Figure 2-1

A	B	A.B
0	0	0
0	1	0
1	0	0
1	1	1

Figure 2-2

A	A'
0	1
1	0

Figure 2-3

**The rule for the complement operator is for verification of postulate 5.**

These rules are exactly the same for as the logical OR, AND, and NOT operations, respectively. It can be shown that the Huntington postulates are applicable for the set  $S = \{0,1\}$  and the two binary operators defined above.

1. Closure is obviously valid, as from the table it is observed that the result of each operation is either 0 or 1 and  $0,1 \in S$ .

2. From the tables, we can see that:

(i)  $0 + 0 = 0 \quad 0 + 1 = 1 + 0 = 1$

(ii)  $1 \cdot 1 = 1 \quad 0 \cdot 1 = 1 \cdot 0 = 0$

which verifies the two identity elements 0 for (+) and 1 for (.) as defined by postulate 2.

3. The commutative laws are confirmed by the symmetry of binary operator tables.

4. The distributive laws of (.) over (+) *i.e.*,  $A \cdot (B+C) = (A \cdot B) + (A \cdot C)$ , and (+) over (.) *i.e.*,  $A + (B \cdot C) = (A+B) \cdot (A+C)$  can be shown to be applicable with the help of the truth tables considering all the possible values of A, B, and C as under.

From the complement table it can be observed that:

(a) Operator (.) over (+)

A	B	C	B+C	A.(B+C)	A.B	A.C	(A.B)+(A.C)
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

Figure 2-4

(b) Operator (+) over (.)

A	B	C	B.C	A+(B.C)	A+B	A+C	(A+B).(A+C)
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Figure 2-5

(c)  $A + A' = 1$ , since  $0 + 0' = 1$  and  $1 + 1' = 1$ .

(d)  $A \cdot A' = 0$ , since  $0 \cdot 0' = 0$  and  $1 \cdot 1' = 0$ .

These confirm postulate 5.

5. Postulate 6 also satisfies two-valued Boolean algebra that has two distinct elements 0 and 1 where 0 is not equal to 1.

## 2.5 BASIC PROPERTIES AND THEOREMS OF BOOLEAN ALGEBRA

### 2.5.1 Principle of Duality

From Huntington postulates, it is evident that they are grouped in pairs as (a) and (b) and every algebraic expression deducible from the postulates of Boolean algebra remains valid if the operators and identity elements are interchanged. This means one expression can be obtained from the other in each pair by interchanging every element *i.e.*, every 0 with 1, every 1 with 0, as well as interchanging the operators *i.e.*, every (+) with (.) and every (.) with (+). This important property of Boolean algebra is called principle of duality.

### 2.5.2 DeMorgan's Theorem

Two theorems that were proposed by DeMorgan play important parts in Boolean algebra.

The first theorem states that the complement of a product is equal to the sum of the complements. That is, if the variables are A and B, then:

$$(A.B)' = A' + B'$$

The second theorem states that the complement of a sum is equal to the product of the complements. In equation form, this can be expressed as:

$$(A + B)' = A' . B'$$

The complements of Boolean logic function or a logic expression may be simplified or expanded by the following steps of DeMorgan's theorem.

- (a) Replace the operator (+) with (.) and (.) with (+) given in the expression.
- (b) Complement each of the terms or variables in the expression.

DeMorgan's theorems are applicable to any number of variables. For three variables A, B, and C, the equations are:

$$(A.B.C)' = A' + B' + C' \text{ and}$$

$$(A + B + C)' = A' . B' . C'$$

### 2.5.3 Other Important Theorems

**Theorem 1(a):**  $A + A = A$

$$\begin{aligned} A + A &= (A + A).1 && \text{by postulate 2(b)} \\ &= (A + A) . (A + A') && \text{by postulate 5} \\ &= A + A.A' \\ &= A + 0 && \text{by postulate 4} \\ &= A && \text{by postulate 2(a)} \end{aligned}$$

**Theorem 1(b):**  $A . A = A$

$$\begin{aligned} A . A &= (A . A) + 0 && \text{by postulate 2(a)} \\ &= (A . A) + (A . A') && \text{by postulate 5} \\ &= A (A + A') \\ &= A . 1 && \text{by postulate 4} \\ &= A && \text{by postulate 2(b)} \end{aligned}$$

**Theorem 2(a):**  $A + 1 = 1$

**Theorem 2(b):**  $A . 0 = 0$

**Theorem 3(a):**  $A + A.B = A$   
 $A + A.B = A . 1 + A.B$  by postulate 2(b)  
 $= A ( 1 + B )$  by postulate 4(a)  
 $= A . 1$  by postulate 2(a)  
 $= A$  by postulate 2(b)

**Theorem 3(b):**  $A ( A + B ) = A$  by duality

The following is the complete list of postulates and theorems useful for two-valued Boolean algebra.

Postulate 2	(a) $A + 0 = A$	(b) $A.1 = A$
Postulate 5	(a) $A + A' = 1$	(b) $A.A' = 0$
Theorem 1	(a) $A + A = A$	(b) $A.A = A$
Theorem 2	(a) $A + 1 = 1$	(b) $A.0 = 0$
Theorem 3, Involution	$(A')' = A$	
Theorem 3, Commutative	(a) $A + B = B + A$	(b) $A.B = B.A$
Theorem 4, Associative	(a) $A + (B + C) = (A + B) + C$	(b) $A.(B.C) = (A.B).C$
Theorem 4, Distributive	(a) $A(B + C) = A.B + A.C$	(b) $A + B.C = (A + B).(A + C)$
Theorem 5, DeMorgan	(a) $(A + B)' = A'.B'$	(b) $(A.B)' = A' + B'$
Theorem 6, Absorption	(a) $A + A.B = A$	(b) $A.(A + B) = A$

Figure 2-6

## 2.6 BOOLEAN FUNCTIONS

Binary variables have two values, either 0 or 1. A Boolean function is an expression formed with binary variables, the two binary operators AND and OR, one unary operator NOT, parentheses and equal sign. The value of a function may be 0 or 1, depending on the values of variables present in the Boolean function or expression. For example, if a Boolean function is expressed algebraically as:

$$F = AB'C$$

then the value of F will be 1, when  $A = 1$ ,  $B = 0$ , and  $C = 1$ . For other values of A, B, C the value of F is 0.

Boolean functions can also be represented by truth tables. A *truth table* is the tabular form of the values of a Boolean function according to the all possible values of its variables. For an  $n$  number of variables,  $2^n$  combinations of 1s and 0s are listed and

one column represents function values according to the different combinations. For example, for three variables the Boolean function  $F = AB + C$  truth table can be written as below in Figure 2.7

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Figure 2-7

A Boolean function from an algebraic expression can be realized to a logic diagram composed of logic gates. Figure 2.8 is an example of a logic diagram realized by the basic gates like AND, OR, and NOT gates.

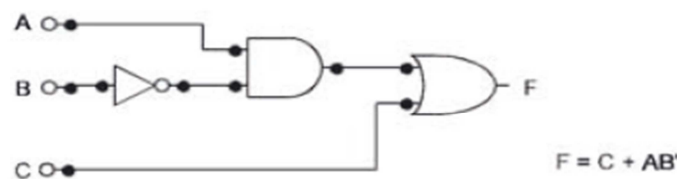


Figure 2-8

## 2.7 SIMPLIFICATION OF BOOLEAN EXPRESSIONS

When a Boolean expression is implemented with logic gates, each literal in the function is designated as input to the gate. The literal may be a primed or unprimed variable. Minimization of the number of literals and the number of terms leads to less complex circuits as well as less number of gates, which should be a designer's aim. There are several methods to minimize the Boolean function. Here, simplification or minimization of complex algebraic expressions will be shown with the help of postulates and theorems of Boolean algebra.

**Example 2.1.** Simplify the Boolean function  $F = AB + BC + B'C$ .

**Solution.**

$$\begin{aligned}
 F &= AB + BC + B'C \\
 &= AB + C(B + B') \\
 &= AB + C
 \end{aligned}$$

**Example 2.2.** Simplify the Boolean function  $F = A + A'B$ .

**Solution.**

$$\begin{aligned} F &= A + A'B \\ &= (A + A')(A + B) \\ &= A + B \end{aligned}$$

**Example 2.3.** Simplify the Boolean function  $F = A'B'C + A'BC + AB'$ .

**Solution.**

$$\begin{aligned} F &= A'B'C + A'BC + AB' \\ &= A'C(B'+B) + AB' \\ &= A'C + AB' \end{aligned}$$

**Example 2.4.** Simplify the Boolean function  $F = AB + (AC)' + AB'C(AB + C)$ .

**Solution.**

$$\begin{aligned} F &= AB + (AC)' + AB'C(AB + C) \\ &= AB + A' + C' + AB'C.AB + AB'C.C \\ &= AB + A' + C' + 0 + AB'C(B.B' = 0 \text{ and } C.C = C) \\ &= ABC + ABC' + A' + C' + AB'C(AB = AB(C + C') = ABC + ABC') \\ &= AC(B + B') + C'(AB + 1) + A' \\ &= AC + C' + A'(B + B' = 1 \text{ and } AB + 1 = 1) \\ &= AC + (AC)' \\ &= 1 \end{aligned}$$

## 2.8 CANONICAL AND STANDARD FORMS

Logical functions are generally expressed in terms of different combinations of logical variables with their true forms as well as the complement forms. Binary logic values obtained by the logical functions and logic variables are in binary form. An arbitrary logic function can be expressed in the following forms.

(i) Sum of the Products (SOP)

(ii) Product of the Sums (POS)

**Product Term.** In Boolean algebra, the logical product of several variables on which a function depends is considered to be a product term. In other words, the AND function is referred to as a product term or standard product. The variables in a product term can be either in true form or in complemented form. For example,  $ABC'$  is a product term.

**Sum Term.** An OR function is referred to as a sum term. The logical sum of several variables on which a function depends is considered to be a sum term. Variables in a sum term can also be either in true form or in complemented form. For example,  $A + B + C'$  is a sum term.

**Sum of Products (SOP).** The logical sum of two or more logical product terms is referred to as a sum of products expression. It is basically an OR operation on AND operated variables. For example,  $Y = AB + BC + AC$  or  $Y = A'B + BC + AC'$  are sum of products expressions.

**Product of Sums (POS).** Similarly, the logical product of two or more logical sum terms is called a product of sums expression. It is an AND operation on OR operated variables. For example,  $Y = (A + B + C)(A + B' + C)(A + B + C')$  or  $Y = (A + B + C)(A' + B' + C')$  are product of sums expressions.

*Standard form.* The standard form of the Boolean function is when it is expressed in sum of the products or product of the sums fashion. The examples stated above, like  $Y = AB + BC + AC$  or  $Y = (A + B + C)(A + B' + C)(A + B + C')$  are the standard forms.

However, Boolean functions are also sometimes expressed in nonstandard forms like  $F = (AB + CD)(A'B' + C'D')$ , which is neither a sum of products form nor a product of sums form. However, the same expression can be converted to a standard form with help of various Boolean properties, as:

$$F = (AB + CD)(A'B' + C'D') = A'B'CD + ABC'D'$$

### 2.8.1 Minterm

A product term containing all  $n$  variables of the function in either true or complemented form is called the minterm. Each minterm is obtained by an AND operation of the variables in their true form or complemented form. For a two-variable function, four different combinations are possible, such as,  $A'B'$ ,  $A'B$ ,  $AB'$ , and  $AB$ . These product terms are called the fundamental products or standard products or minterms. In the minterm, a variable will possess the value 1 if it is in true or uncomplemented form, whereas, it contains the value 0 if it is in complemented form. For three variables function, eight minterms are possible as listed in the following table in Figure 2.9.

A	B	C	Minterm
0	0	0	$A'B'C'$
0	0	1	$A'B'C$
0	1	0	$A'BC'$
0	1	1	$A'BC$
1	0	0	$AB'C'$
1	0	1	$AB'C$
1	1	0	$ABC'$
1	1	1	$ABC$

Figure 2-9

So, if the number of variables is  $n$ , then the possible number of minterms is  $2^n$ . The main property of a minterm is that it possesses the value of 1 for only one combination of  $n$  input variables and the rest of the  $2^n - 1$  combinations have the logic value of 0. This means, for the above three variables example, if  $A = 0$ ,  $B = 1$ ,  $C = 1$  i.e., for input combination of 011, there is only one combination  $A'BC$  that has the value 1, the rest of the seven combinations have the value 0.

*Canonical Sum of Product Expression.* When a Boolean function is expressed as the logical sum of all the minterms from the rows of a truth table, for which the value of the function is 1, it is referred to as the *canonical sum of product expression*. The

same can be expressed in a compact form by listing the corresponding decimal-equivalent codes of the minterms containing a function value of 1. For example, if the canonical sum of product form of a three-variable logic function  $F$  has the minterms  $A'BC$ ,  $AB'C$ , and  $ABC'$ , this can be expressed as the sum of the decimal codes corresponding to these minterms as below.

$$\begin{aligned} F(A,B,C) &= (3,5,6) \\ &= m_3 + m_5 + m_6 \\ &= A'BC + AB'C + ABC' \end{aligned}$$

where  $\Sigma(3,5,6)$  represents the summation of minterms corresponding to decimal codes 3, 5, and 6.

The canonical sum of products form of a logic function can be obtained by using the following procedure:

1. Check each term in the given logic function. Retain if it is a minterm, continue to examine the next term in the same manner.
2. Examine for the variables that are missing in each product which is not a minterm. If the missing variable in the minterm is  $X$ , multiply that minterm with  $(X+X')$ .
3. Multiply all the products and discard the redundant terms.

Here are some examples to explain the above procedure.

**Example 2.6.** Obtain the canonical sum of product form of the following function:

$$F(A, B) = A + B$$

**Solution.** The given function contains two variables  $A$  and  $B$ . The variable  $B$  is missing from the first term of the expression and the variable  $A$  is missing from the second term of the expression. Therefore, the first term is to be multiplied by  $(B + B')$  and the second term is to be multiplied by  $(A + A')$  as demonstrated below.

$$\begin{aligned} F(A, B) &= A + B \\ &= A.1 + B.1 \\ &= A(B + B') + B(A + A') \\ &= AB + AB' + AB + A'B \\ &= AB + AB' + A'B \quad (\text{as } AB + AB = AB) \end{aligned}$$

Hence the canonical sum of the product expression of the given function is

$$F(A, B) = AB + AB' + A'B.$$

**Example 2.7.** Obtain the canonical sum of product form of the following function.

$$F(A, B, C) = A + BC$$

**Solution.** Here neither the first term nor the second term is minterm. The given function contains three variables  $A$ ,  $B$ , and  $C$ . The variables  $B$  and  $C$  are missing from the first term of the expression and the variable  $A$  is missing from the second term of the expression. Therefore, the first term is to be multiplied by  $(B + B')$  and  $(C + C')$ . The second term is to be multiplied by  $(A + A')$ . This is demonstrated below.

$$\begin{aligned} F(A, B, C) &= A + BC \\ &= A(B + B')(C + C') + BC(A + A') \\ &= (AB + AB')(C + C') + ABC + A'BC \\ &= ABC + AB'C + ABC' + AB'C' + ABC + A'BC \end{aligned}$$

$$= ABC + AB'C + ABC' + AB'C' + A'BC \text{ (as } ABC + ABC = ABC)$$

Hence the canonical sum of the product expression of the given function is

$$F(A, B, C) = ABC + AB'C + ABC' + AB'C' + A'BC.$$

### 2.8.2 Maxterm

A sum term containing all  $n$  variables of the function in either true or complemented form is called the maxterm. Each maxterm is obtained by an OR operation of the variables in their true form or complemented form. Four different combinations are possible for a two-variable function, such as,  $A' + B'$ ,  $A' + B$ ,  $A + B'$ , and  $A + B$ . These sum terms are called the standard sums or maxterms. Note that, in the maxterm, a variable will possess the value 0, if it is in true or uncomplemented form, whereas, it contains the value 1, if it is in complemented form. Like minterms, for a three-variable function, eight maxterms are also possible as listed in the following table in Figure 2.10.

A	B	C	Maxterm
0	0	0	$A + B + C$
0	0	1	$A + B + C'$
0	1	0	$A + B' + C$
0	1	1	$A + B' + C'$
1	0	0	$A' + B + C$
1	0	1	$A' + B + C'$
1	1	0	$A' + B' + C$
1	1	1	$A' + B' + C'$

Figure 2-10

So, if the number of variables is  $n$ , then the possible number of maxterms is  $2^n$ . The main property of a maxterm is that it possesses the value of 0 for only one combination of  $n$  input variables and the rest of the  $2^n - 1$  combinations have the logic value of 1. This means, for the above three variables example, if  $A = 1$ ,  $B = 1$ ,  $C = 0$  i.e., for input combination of 110, there is only one combination  $A' + B' + C$  that has the value 0, the rest of the seven combinations have the value 1.

*Canonical Product of Sum Expression.* When a Boolean function is expressed as the logical product of all the maxterms from the rows of a truth table, for which the value of the function is 0, it is referred to as the *canonical product of sum expression*. The same can be expressed in a compact form by listing the corresponding decimal equivalent codes of the maxterms containing a function value of 0. For example, if the canonical product of sums form of a three-variable logic function  $F$  has the maxterms  $A + B + C$ ,  $A + B' + C$ , and  $A' + B + C'$ , this can be expressed as the product of the decimal codes corresponding to these maxterms as below,

$$\begin{aligned}
F(A,B,C) &= \Pi(0,2,5) \\
&= M_0 M_2 M_5 \\
&= (A + B + C)(A + B' + C)(A' + B + C')
\end{aligned}$$

where  $\Pi(0,2,5)$  represents the product of maxterms corresponding to decimal codes 0, 2, and 5.

The canonical product of sums form of a logic function can be obtained by using the following procedure.

1. Check each term in the given logic function. Retain it if it is a maxterm, continue to examine the next term in the same manner.

2. Examine for the variables that are missing in each sum term that is not a maxterm. If the missing variable in the maxterm is X, add that maxterm with  $(X.X')$ .

3. Expand the expression using the properties and postulates as described earlier and discard the redundant terms.

Some examples are given here to explain the above procedure.

**Example 2.8.** Obtain the canonical product of the sum form of the following function.

$$F(A, B, C) = (A + B')(B + C)(A + C')$$

**Solution.** In the above three-variable expression, C is missing from the first term, A is missing from the second term, and B is missing from the third term. Therefore,  $CC'$  is to be added with first term,  $AA'$  is to be added with the second, and  $BB'$  is to be added with the third term. This is shown below.

$$\begin{aligned}
F(A, B, C) &= (A + B')(B + C)(A + C') \\
&= (A + B' + 0)(B + C + 0)(A + C' + 0) \\
&= (A + B' + CC')(B + C + AA')(A + C' + BB') \\
&= (A + B' + C)(A + B' + C')(A + B + C)(A' + B + C)(A + B + C')(A + B' + C') \\
&[\text{using the distributive property, as } X + YZ = (X + Y)(X + Z)] \\
&= (A + B' + C)(A + B' + C')(A + B + C)(A' + B + C)(A + B + C') \\
&[\text{as } (A + B' + C)(A + B' + C') = A + B' + C']
\end{aligned}$$

Hence the canonical product of the sum expression for the given function is

$$F(A, B, C) = (A + B' + C)(A + B' + C')(A + B + C)(A' + B + C)(A + B + C')$$

**Example 2.9.** Obtain the canonical product of the sum form of the following function.

$$F(A, B, C) = A + B'C$$

**Solution.** In the above three-variable expression, the function is given at sum of the product form. First, the function needs to be changed to product of the sum form by applying the distributive law as shown below.

$$\begin{aligned}
F(A, B, C) &= A + B'C \\
&= (A + B')(A + C)
\end{aligned}$$

Now, in the above expression, C is missing from the first term and B is missing from the second term. Hence  $CC'$  is to be added with the first term and  $BB'$  is to be added with the second term as shown below.

$$\begin{aligned}
F(A, B, C) &= (A + B')(A + C) \\
&= (A + B' + CC')(A + C + BB')
\end{aligned}$$

$$\begin{aligned}
&= (A + B' + C) (A + B' + C') (A + B + C) (A + B' + C) \\
&[\text{using the distributive property, as } X + YZ = (X + Y) (X + Z)] \\
&= (A + B' + C) (A + B' + C') (A + B + C) \\
&[\text{as } (A + B' + C) (A + B' + C) = A + B' + C]
\end{aligned}$$

Hence the canonical product of the sum expression for the given function is  
 $F(A, B, C) = (A + B' + C) (A + B' + C') (A + B + C)$ .

### 2.8.3 Deriving a Sum of Products (SOP) Expression from a Truth Table

The sum of products (SOP) expression of a Boolean function can be obtained from its truth table summing or performing OR operation of the product terms corresponding to the combinations containing a function value of 1. In the product terms the input variables appear either in true (uncomplemented) form if it contains the value 1, or in complemented form if it possesses the value 0.

Now, consider the following truth table in Figure 2.11, for a three-input function Y. Here the output Y value is 1 for the input conditions of 010, 100, 101, and 110, and their corresponding product terms are  $A'BC'$ ,  $AB'C'$ ,  $AB'C$ , and  $ABC'$  respectively.

Inputs			Output Y	Product terms	Sum terms
A	B	C			
0	0	0	0		$A + B + C$
0	0	1	0		$A + B + C'$
0	1	0	1	$A'BC'$	
0	1	1	0		$A + B' + C'$
1	0	0	1	$AB'C'$	
1	0	1	1	$AB'C$	
1	1	0	1	$ABC'$	
1	1	1	0		$A' + B' + C'$

Figure 2-11

The final sum of products expression (SOP) for the output Y is derived by summing or performing an OR operation of the four product terms as shown below.

$$Y = A'BC' + AB'C' + AB'C + ABC'$$

In general, the procedure of deriving the output expression in SOP form from a truth table can be summarized as below.

1. Form a product term for each input combination in the table, containing an output value of 1.
2. Each product term consists of its input variables in either true form or complemented form. If the input variable is 0, it appears in complemented form and if the input variable is 1, it appears in true form.
3. To obtain the final SOP expression of the output, all the product terms are OR operated.

### 2.8.4 Deriving a Product of Sums (POS) Expression from a Truth Table

As explained above, the product of sums (POS) expression of a Boolean function can also be obtained from its truth table by a similar procedure. Here, an AND operation is performed on the sum terms corresponding to the combinations containing a function value of 0. In the sum terms the input variables appear either in true (uncomplemented) form if it contains the value 0, or in complemented form if it possesses the value 1.

Now, consider the same truth table as shown in Figure 2.11, for a three-input function Y. Here the output Y value is 0 for the input conditions of 000, 001, 011, and 111, and their corresponding product terms are  $A + B + C$ ,  $A + B + C'$ ,  $A + B' + C'$ , and  $A' + B' + C'$  respectively.

So now, the final product of sums expression (POS) for the output Y is derived by performing an AND operation of the four sum terms as shown below.

$$Y = (A + B + C) (A + B + C') (A + B' + C') (A' + B' + C')$$

In general, the procedure of deriving the output expression in POS form from a truth table can be summarized as below.

1. Form a sum term for each input combination in the table, containing an output value of 0.
2. Each product term consists of its input variables in either true form or complemented form. If the input variable is 1, it appears in complemented form and if the input variable is 0, it appears in true form.
3. To obtain the final POS expression of the output, all the sum terms are AND operated.

### 2.9.5 Conversion between Canonical Forms

From the above example, it may be noted that the complement of a function expressed as the sum of products (SOP) equals to the sum of products or sum of the minterms which are missing from the original function. This is because the original function is expressed by those minterms that make the function equal to 1, while its complement is 1 for those minterms whose values are 0. According to the truth table given in Figure 2.11:

$$\begin{aligned} F(A,B,C) &= \Sigma (2,4,5,6) \\ &= m_2 + m_4 + m_5 + m_6 \\ &= A'BC' + AB'C' + AB'C + ABC' \end{aligned}$$

This has the complement that can be expressed as

$$\begin{aligned} F'(A,B,C) &= (0,1,3,7) \\ &= m_0 + m_1 + m_3 + m_7 \end{aligned}$$

Now, if we take complement of  $F'$  by DeMorgan's theorem, we obtain F as

$$\begin{aligned} F(A,B,C) &= (m_0 + m_1 + m_3 + m_7)' \\ &= m_0' m_1' m_3' m_7' \\ &= M_0 M_1 M_3 M_7 \\ &= \Pi(0,1,3,7) \\ &= (A + B + C)(A + B + C')(A + B' + C')(A' + B' + C') \end{aligned}$$

The last conversion follows from the definition of minterms and maxterms as shown in the tables in Figures 2.9 and 2.10. It can be clearly noted that the following relation holds true  $m'j = Mj$ .

That is, the maxterm with subscript  $j$  is a complement of the minterm with the same subscript  $j$ , and vice versa.

This example demonstrates the conversion between a function expressed in sum of products (SOP) and its equivalent in product of maxterms. A similar example can show the conversion between the product of sums (POS) and its equivalent sum of minterms. In general, to convert from one canonical form to other canonical form, it is required to interchange the symbols  $\Sigma$  and  $\pi$ , and list the numbers which are missing from the original form.

Note that, to find the missing terms, the total  $2^n$  number of minterms or maxterms must be realized, where  $n$  is the number of variables in the function.

## ***2.9 DIGITAL LOGIC GATES***

As Boolean functions are expressed in terms of AND, OR, and NOT operations, it is easier to implement the Boolean functions with these basic types of gates. However, for all practical purposes, it is possible to construct other types of logic gates. The following factors are to be considered for construction of other types of gates.




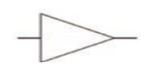
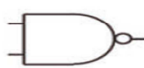



Name	Graphic Symbol	Algebraic Function	Truth Table		
AND		$F = AB$	A	B	F
			0	0	0
			0	1	0
			1	0	0
			1	1	1
OR		$F = A + B$	A	B	F
			0	0	0
			0	1	1
			1	0	1
			1	1	1
Inverter or NOT		$F = A'$	A	F	
			0	1	
			1	0	
Buffer		$F = A$	A	F	
			0	0	
			1	1	
NAND		$F = (AB)'$	A	B	F
			0	0	1
			0	1	1
			1	0	1
			1	1	0
NOR		$F = (A + B)'$	A	B	F
			0	0	1
			0	1	0
			1	0	0
			1	1	0
Exclusive-OR (XOR)		$F = AB' + A'B$ $= A \oplus B$	A	B	F
			0	0	0
			0	1	1
			1	0	1
			1	1	0
Equivalence Or Exclusive-NOR (XNOR)		$F = AB + A'B'$	A	B	F
			0	0	1
			0	1	0
			1	0	0
			1	1	1

Figure 2-12

1. The feasibility and economy of producing the gate with physical parameters.
2. The possibility of extending to more than two inputs.
3. The basic properties of the binary operator such as commutability and associability.
4. The ability of the gate to implement the Boolean functions alone or in conjunction with other gates.

There are eight functions—Transfer (or buffer), Complement, AND, OR, NAND, NOR, Exclusive-OR (XOR), and Equivalence (XNOR) that may be considered to be standard gates in digital design.

The transfer or buffer and complement or inverter or NOT gates are unary gates, *i.e.*, they have single input, while other logic gates have two or more inputs.

### 2.9.1 Extension to Multiple Inputs

A gate can be extended to have multiple inputs if its binary operation is commutative and associative. AND and OR gates are both commutative and associative.

For the AND function,  $AB = BA$  -commutative

and

$(AB)C = A(BC) = ABC$ . -associative

For the OR function,  $A + B = B + A$  -commutative

and

$(A + B) + C = A + (B + C)$ . -associative

These indicate that the gate inputs can be interchanged and these functions can be extended to three or more variables very simply as shown in Figures 2.13(a) and 2.13(b).



Figure 2-13(a)



Figure 2-13(b)

The NAND and NOR functions are the complements of AND and OR functions respectively. They are commutative, but not associative. So these functions can not be extended to multiple input variables very simply. However, these gates can be extended to multiple inputs with slightly modified functions as shown in Figures 2.14(a) and 2.14(b) below.

For NAND function,  $(AB)' = (BA)'$ . -commutative  
 But,  $((AB)'C)' \neq (A(BC)')'$ . -does not follow associative property.

As  $((AB)'C)' = (AB) + C'$  and  
 $(A(BC)')' = A' + BC$ .

Similarly, for NOR function,  $((A + B)' + C)' \neq (A + (B + C)')'$ .

As,  $((A + B)' + C)' = (A + B)C' = AC' + BC'$ .

And  $(A + (B + C)')' = A'(B + C) = A'B + A'C$ .

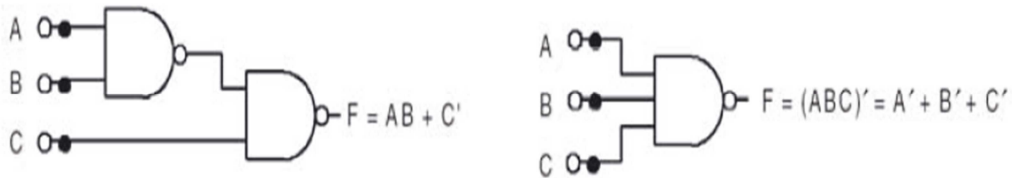


Figure 2-14(a)

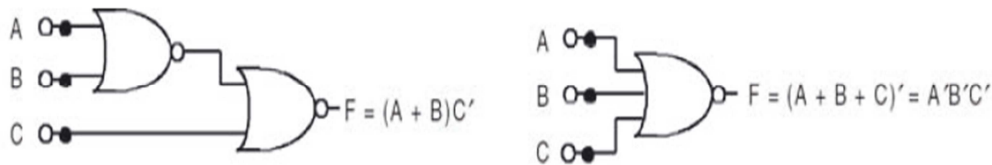


Figure 2-14(b)

The Exclusive-OR gates and equivalence gates both possess commutative and associative properties, and they can be extended to multiple input variables. For a multiple-input Ex-OR (XOR) gate output is low when even numbers of 1s are applied to the inputs, and when the number of 1s is odd the output is logic 1. Equivalence gate or XNOR gate is equivalent to XOR gate followed by NOT gate and hence its logic behavior is opposite to the XOR gate. However, multiple-input exclusive-OR and equivalence gates are uncommon in practice. Figures 2.15(a) and 2.15(b) describe the extension to multiple-input exclusive-OR and equivalence gates.



Figure 2-15(a)

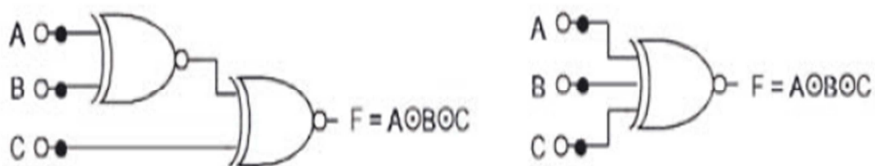
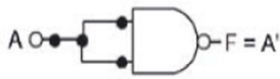


Figure 2-15(b)

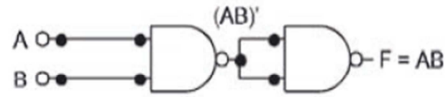
### 2.9.2 Universal Gates

NAND gates and NOR gates are called *universal gates* or *universal building blocks*, as any type of gates or logic functions can be implemented by these gates. Figures 2.16(a)-(e) show how various logic functions can be realized by NAND gates and Figures 2.17(a)-(d) show the realization of various logic gates by NOR gates.



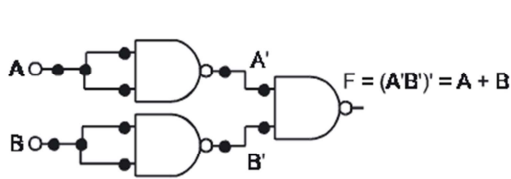
NOT function:  $F = A'$

Figure 2-16(a)



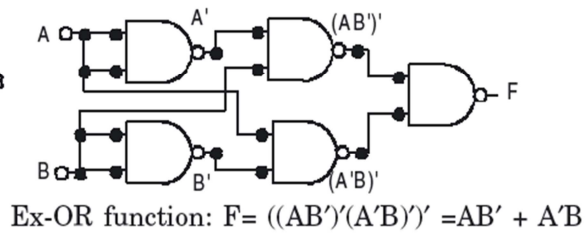
AND function:  $F = AB$

Figure 2-16(b)



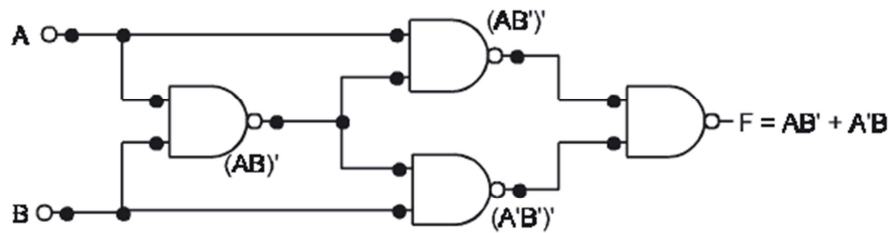
OR function:  $F = A + B$

Figure 2-16(c)



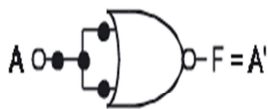
Ex-OR function:  $F = ((AB)'(A'B)')' = AB' + A'B$

Figure 2-16(d)



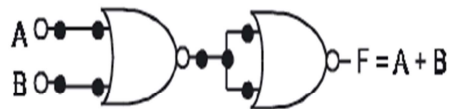
Ex-OR gate with reduced number of NAND gates

Figure 2-16(e)



NOT function:  $F = A'$

Figure 2-17(a)



OR function:  $F = A + B$

Figure 2-17(b)

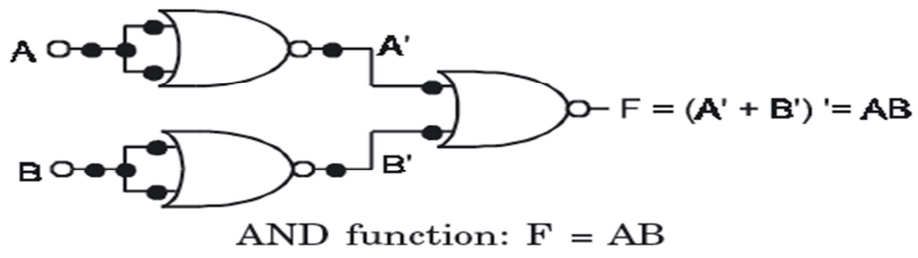


Figure 2-17(c)

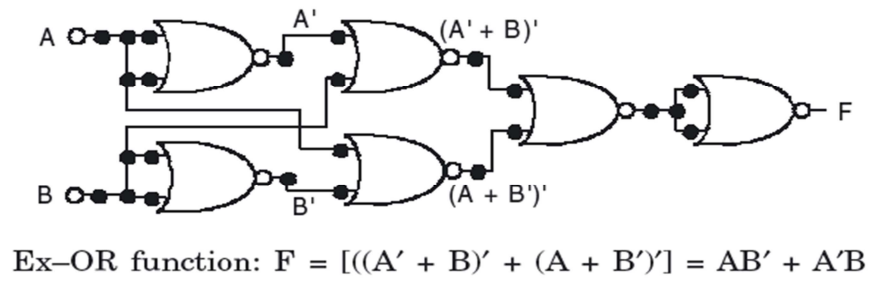


Figure 2-17(d)