4.2 Zero of Polynomial Functions

Factor Theorem
Rational Zeros Theorem
Number of Zeros
Conjugate Zeros Theorem
Finding Zeros of a Polynomial Function

Factor Theorem

The polynomial $x - k$ is a factor of the polynomial $f(x)$ if and only if $f(k) = 0$.

Example 2 FACTORING A POLYNOMIAL GIVEN A ZERO

Factor the following into linear factors if –3 is a zero of f. $f(x) = 6x^3 + 19x^2 + 2x - 3$

Solution Since –3 is a zero of f, $x - (-3) = x + 3$ is a factor.

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 2 & -3 \\ & & -18 & -3 & 3 \\ \hline & 6 & 1 & -1 & 0 \end{array}$$

Use synthetic division to divide $f(x)$ by $x + 3$.

The quotient is $6x^2 + x - 1$.

Example 2 FACTORING A POLYNOMIAL GIVEN A ZERO

Factor the following into linear factors if –3 is a zero of f. $f(x)=6x^3+19x^2+2x-3$

Solution $x-(-3)=x+3$ is a factor.

The quotient is $6x^2+x-1$, so

$$f(x)=(x+3)\underbrace{(6x^2+x-1)}$$

$$f(x)=(x+3)(2x+1)(3x-1).$$ Factor $6x^2+x-1$.

These factors are all linear.

Rational Zeros Theorem

If $\frac{p}{q}$ is a rational number written in lowest terms, and if $\frac{p}{q}$ is a zero of f, a polynomial function with integer coefficients, then p is a factor of the constant term and q is a factor of the leading coefficient.

Example 3 USING THE RATIONAL ZERO THEOREM

Do the following for the polynomial function defined by $f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2.$

a. List all possible rational zeros.

Solution For a rational number $\frac{p}{q}$ to be zero, p must be a factor of $a_0 = 2$ and q must be a factor of $a_4 = 6$. Thus, p can be ± 1 or ± 2, and q can be ± 1, ± 2, ± 3, or ± 6. The possible rational zeros, $\frac{p}{q}$ are,

$$\pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3}.$$

Example 3 USING THE RATIONAL ZERO THEOREM

Do the following for the polynomial function defined by $f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2.$

b. Find all rational zeros and factor $f(x)$ into linear factors.

Solution Use the remainder theorem to show that 1 is a zero.

Use "trial and error" to find zeros.

$$\begin{array}{r|rrrrr} 1 & 6 & 7 & -12 & -3 & 2 \\ & & 6 & 13 & 1 & -2 \\ \hline & 6 & 13 & 1 & -2 & 0 \end{array} \quad \longleftarrow \quad f(1) = 0$$

The 0 remainder shows that 1 is a zero. The quotient is $6x^3 + 13x^2 + x - 4$, so $f(x) = (x - 1)(6x^3 + 13x^2 + x - 2)$.

Example 3 USING THE RATIONAL ZERO THEOREM

Do the following for the polynomial function defined by $f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$.

b. Find all rational zeros and factor $f(x)$ into linear equations.

Solution Now, use the quotient polynomial and synthetic division to find that –2 is a zero.

$$\begin{array}{r|rrrr} -2 & 6 & 13 & 1 & -2 \\ & & -12 & -2 & 2 \\ \hline & 6 & 1 & -1 & 0 \end{array} \quad \longleftarrow \ f(-2) = 0$$

The new quotient polynomial is $6x^2 + x - 1$.

Therefore, $f(x)$ can now be factored.

Example 3 **USING THE RATIONAL ZERO THEOREM**

Do the following for the polynomial function defined by $f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2.$

b. Find all rational zeros and factor $f(x)$ into linear equations.

Solution

$$f(x) = (x-1)(x+2)(6x^2 + x - 1)$$

$$= (x-1)(x+2)(3x-1)(2x+1).$$

Example 3 USING THE RATIONAL ZERO THEOREM

Do the following for the polynomial function defined by $f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$.

b. Find all rational zeros and factor $f(x)$ into linear equations.

Solution Setting $3x - 1 = 0$ and $2x + 1 = 0$ yields the zeros ⅓ and –½. In summary the rational zeros are 1, –2, ⅓, –½, and the linear factorization of $f(x)$ is

$$f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$$

$$= (x - 1)(x + 2)(3x - 1)(2x + 1).$$

Check by multiplying these factors.

Example 4

FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function f defined by a polynomial of degree 3 that satisfies the given conditions.

a. Zeros of –1, 2, and 4; $f(1) = 3$

Solution These three zeros give $x - (-1) = x + 1$, $x - 2$, and $x - 4$ as factors of $f(x)$. Since $f(x)$ is to be of degree 3, these are the only possible factors by the number of zeros theorem. Therefore, $f(x)$ has the form

$$f(x) = a(x+1)(x-2)(x-4)$$

for some real number a.

Example 4 FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function f defined by a polynomial of degree 3 that satisfies the given conditions.

a. Zeros of –1, 2, and 4; $f(1) = 3$

Solution To find *a*, use the fact that $f(1) = 3$.

$$f(1) = a(1+1)(1-2)(1-4)$$ Let $x = 1$.

$$3 = a(2)(-1)(-3)$$ $f(1) = 3$

$$3 = 6a$$ Solve for *a*.

$$a = \frac{1}{2}$$

Example 4

FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function f defined by a polynomial of degree 3 that satisfies the given conditions.

a. Zeros of –1, 2, and 4; $f(1) = 3$

Solution Thus,

$$f(x) = \frac{1}{2}(x+1)(x-2)(x-4),$$

or

$$f(x) = \frac{1}{2}x^3 - \frac{5}{2}x^2 + x + 4.$$ Multiply.

Example 4

FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function f defined by a polynomial of degree 3 that satisfies the given conditions.

b. –2 is a zero of multiplicity 3; $f(–1) = 4$

Solution The polynomial function defined by $f(x)$ has the form

$$f(x) = a(x+2)(x+2)(x+2)$$

$$= a(x+2)^3.$$

Example 4

FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function f defined by a polynomial of degree 3 that satisfies the given conditions.

b. –2 is a zero of multiplicity 3; $f(-1) = 4$

Solution Since $f(-1) = 4$,

$$f(-1) = a(-1+2)^3$$

$$4 = a(1)^3$$

$$a = 4,$$

Remember:
$(x + 2)^3 \neq x^3 + 2^3$

and $f(x) = 4(x+2)^3 = 4x^3 + 24x^2 + 48x + 32.$