3.2.2 Bingham plastic and yield-pseudoplastic fluids

A fluid with a yield stress will flow only if the applied stress (proportional to pressure gradient) exceeds the yield stress. There will be a solid plug-like core flowing in the middle of the pipe where $|\tau_{rz}|$ is less than the yield stress, as shown schematically in Figure 3.4. Its radius, R_p, will depend upon the magnitude of the yield stress and on the wall shear stress. From equation (3.2),

$$\frac{\tau_0^B}{\tau_w} = \frac{R_p}{R} \tag{3.9}$$

where τ_w is the shear stress at the wall of the pipe.

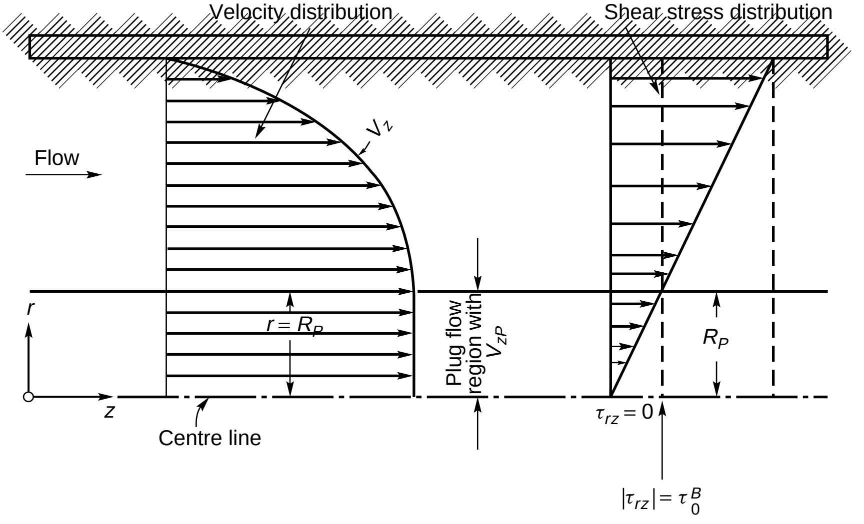

Figure 3.4 *Schematic velocity distribution for laminar flow of a Bingham plastic fluid in a pipe*

In the annular area $R_p < r < R$, the velocity will gradually decrease from the constant plug velocity to zero at the pipe wall. The expression for this velocity distribution will now be derived.

For the region $R_p < r < R$, the value of shear stress will be greater than the yield stress of the fluid, and the Bingham fluid model for pipe flow is given by (equation (1.16) in Chapter 1):

$$\tau_{rz} = \tau_0^B + \mu_B \left(-\frac{\mathrm{d}V_z}{\mathrm{d}r}\right) \tag{3.10}$$

Now combining equations (3.2) and (3.10) followed by integration yields the following expression for the velocity distribution

$$V_z = -\frac{1}{\mu_B}\left(\frac{-\Delta p}{L}\;\frac{r^2}{2} - r\tau_0^B\right) + \text{constant}$$

At the walls of the pipe (i.e. when $r = R$), the velocity V_z must be zero to satisfy the condition of no-slip. Substituting the value $V_z = 0$, when $r = R$:

$$\text{constant} = \frac{1}{\mu_B}\left(\left(\frac{-\Delta p}{L}\right)\frac{R^2}{2} - R\tau_0^B\right)$$

and, therefore:

$$V_z = \left(\frac{-\Delta p}{L}\right)\frac{R^2}{4\mu_B}\left(1 - \frac{r^2}{R^2}\right) - \frac{\tau_0^B}{\mu_B}R\left(1 - \frac{r}{R}\right) \tag{3.11}$$

Clearly, equation (3.11) is applicable only when $|\tau_{rz}| > \tau_0^B$ and $r \geq R_p$. The corresponding velocity, V_{zp}, in the plug region ($0 \leq r \leq R_p$) is obtained by substituting $r = R_p$ in equation (3.11) to give:

$$V_{zp} = \left(\frac{-\Delta p}{L}\right)\frac{R^2}{4\mu_B}\left(1 - \frac{R_p}{R}\right)^2 \qquad 0 \leq r \leq R_p \tag{3.12}$$

The corresponding expression for the volumetric flow rate, Q, is obtained by evaluating the integral

$$Q = \int_0^R 2\pi r V_z \,\mathrm{d}r = \int_0^{R_p} 2\pi r V_{zp} \,\mathrm{d}r + \int_{R_p}^R 2\pi r V_z \,\mathrm{d}r$$

Substitution from equations (3.11) and (3.12), and integration yields for Q:

$$Q = \frac{\pi R^4}{8\mu_B}\left(\frac{-\Delta p}{L}\right)\left(1 - \frac{4}{3}\phi + \frac{1}{3}\phi^4\right) \tag{3.13}$$

where $\phi = \tau_0^B/\tau_w$.

It is useful to re-write equation (3.13) in a dimensionless form as:

$$f = \frac{16}{\mathrm{Re}_B}\left(1 + \frac{1}{6}\frac{He}{\mathrm{Re}_B} - \frac{1}{3}\frac{He^4}{f^3\mathrm{Re}_B^7}\right) \tag{3.13a}$$

where f is the usual friction factor defined as $(-\Delta p/2\rho u^2 \cdot D/L)$; Re_B is the Reynolds number ($= \rho VD/\mu_B$) and He is the Hedström number defined as $(\rho D^2\tau_0^B/\mu_B^2)$ [Hedström, 1952]. It should be noted that equation (3.13) is implicit in pressure gradient (because τ_w, and hence ϕ, is a function of pressure gradient) and therefore, for a specified flow rate, an iterative method is needed to evaluate the pressure drop.

This analysis can readily be extended to the laminar flow of Herschel–Bulkley model fluids (equation 1.17), and the resulting final expressions for the point velocity and volumetric flow rate are [Skelland, 1967; Govier and Aziz, 1982; Bird *et al*., 1983, 1987]:

$$V_z = \frac{nR}{(n+1)}\left(\frac{\tau_w}{m}\right)^{1/n}\left\{(1-\phi)^{(n+1)/n} - \left(\frac{r}{R}-\phi\right)^{(n+1)/n}\right\} \tag{3.14a}$$

and

$$Q = \pi R^3 n\left(\frac{\tau_w}{m}\right)^{1/n}(1-\phi)^{(n+1)/n}\left[\frac{(1-\phi)^2}{3n+1}+\frac{2\phi(1-\phi)}{2n+1}+\frac{\phi^2}{n+1}\right] \tag{3.14b}$$

where ϕ is now the ratio (τ_0^H/τ_w). These expressions are also implicit in pressure gradient.

Example 3.2

The rheological properties of a china clay suspension can be approximated by either a power-law or a Bingham plastic model over the shear rate range 10 to 100 s^{-1}. If the yield stress is 15 Pa and the plastic viscosity is 150 mPa·s, what will be the approximate values of the power-law consistency coefficient and flow behaviour index?

Estimate the pressure drop when this suspension is flowing under laminar conditions in a pipe of 40 mm diameter and 200 m long, when the centre-line velocity is 0.6 m/s, according to the Bingham plastic model? Calculate the centre-line velocity for this pressure drop for the power-law model.

Solution

Using the Bingham plastic model,

$$\tau_{rz} = \tau_0^B + \mu_B(-\mathrm{d}V_z/\mathrm{d}r)$$

when $-\mathrm{d}V_z/\mathrm{d}r = 10\,\mathrm{s}^{-1}$, $\tau_{rz} = 15 + 150\times10^{-3}\times10 = 16.5\,\mathrm{Pa}$

$-\mathrm{d}V_z/\mathrm{d}r = 100\,\mathrm{s}^{-1}$, $\tau_{rz} = 15 + 150\times10^{-3}\times100 = 30\,\mathrm{Pa}$

Now using the power-law model,

$$\tau_{rz} = m\left(-\frac{\mathrm{d}V_z}{\mathrm{d}r}\right)^n$$

Substituting the values of τ_{rz} and $(-\mathrm{d}V_z/\mathrm{d}r)$:

$16.5 = m(10)^n$, and

$30 = m(100)^n$

Solving for m and n gives:

$n = 0.26, \quad m = 9.08\,\mathrm{Pa{\cdot}s}^n$

For a Bingham plastic fluid, equation (3.12) gives:

$$V_{z\,\max} = V_{zp} = 0.6 = \left(\frac{-\Delta p}{L}\right. \left(\frac{(20 \times 10^{-3})^2}{4 \times 0.15}\right. \left(1 - \frac{R_p}{R}\right.^{2}$$

Substitution for (R_p/R) from equation (3.9) and writing the wall shear stress in terms of pressure gradient gives:

$$0.6 = \left(\frac{-\Delta p}{L}\right. \left(\frac{(20 \times 10^{-3})^2}{4 \times 0.15}\right. \quad 1 - \frac{15}{\dfrac{20 \times 10^{-3}}{2}\left(\dfrac{-\Delta p}{L}\right.}^{2}$$

A trial and error procedure leads to

$$\frac{-\Delta p}{L} = 3200\,\text{Pa/m}$$

and therefore the total pressure drop over the pipe length, $-\Delta p = 3200 \times 200 =$ 640 kPa

The centre-line velocity according to the power-law model is given by equation (3.4) with $r = 0$, i.e.

$$\begin{aligned} V_{z,\max} &= \left(\frac{n}{n+1}\right. \left(-\frac{\Delta p}{mL} \cdot \frac{R}{2}\right.^{1/n} \cdot R \\ &= \left(\frac{0.26}{0.26+1}\right. \left(\frac{3200 \times 20 \times 10^{-3}}{9.08 \times 2}\right.^{1/0.26} \times 20 \times 10^{-3} \\ &= 0.52\,\text{m/s} \end{aligned}$$

One can easily see that the plug like motion occurs across a substantial portion of the cross-section as $R_p = 0.47\,$R.

Before concluding this section, it is appropriate to mention here that one can establish similar $Q\!-\Delta p$ relations for the other commonly used viscosity models in an identical manner. A summary of such relations can be found in standard textbooks [Skelland, 1967; Govier and Aziz, 1982].