3.2.3 *Average kinetic energy of fluid*

,In order to obtain the kinetic energy correction factor
for insertion in the mechanical energy balance, it is necessary to evaluate the ,
average kinetic energy per unit mass in terms of the average velocity of flow. The calcula-tion procedure is exactly similar to that used for Newtonian fluids, (e.g. see .([[Coulson and Richardson, 1999

Average kinetic energy/unit mass

$$= \frac{\frac{1}{2}V_z^2\,\mathrm{d}\dot{m}}{\mathrm{d}\dot{m}} = \frac{\int_0^R \frac{1}{2}V_z^2 2\pi r V_z \rho \mathrm{d}r}{\int_0^R 2\pi r V_z \rho \mathrm{d}r} \qquad (3.15a)$$

$$= \frac{V^2}{2\alpha} \qquad (3.15b)$$

where α is a kinetic energy correction factor to take account of the non-uniform velocity over the cross-section. For power-law fluids, substitution for V_z from equation (3.7a) into equation (3.15a) and integration gives

$$\alpha = \frac{(2n+1)(5n+3)}{3(3n+1)^2} \qquad (3.16)$$

The corresponding expression for a Bingham plastic is cumbersome. However, Metzner [1956] gives a simple expression for α which is accurate to within 2.5%:

$$\alpha = \frac{1}{2-\phi} \qquad (3.17)$$

Again, both equations (3.16) and (3.17) reduce to $\alpha = 1/2$ for Newtonian fluid behaviour. Note that as the degree of shear-thinning increases, i.e. the value of n decreases, the kinetic energy correction factor approaches unity at $n = 0$ as would be expected, as all the fluid is flowing at the same velocity (Figure 3.3). For shear-thickening fluids, on the other hand, it attains a limiting value of 0.37 for the infinite degree of shear-thickening behaviour ($n = \infty$).

3.2.4 Generalised approach for laminar flow of time-independent fluids

Approach used in section 3.2 for power-law and Bingham plastic model fluids can be extended to other fluid models. Even if the relationship between shear stress and shear rate is not known exactly, it is possible to use the following approach to the problem. It depends upon the fact that the shear stress distribution over the pipe cross-section is not a function of the fluid rheology and is given simply by equation (3.2), which can be re-written in terms of the wall shear stress, i.e.

$$\frac{\tau_{rz}}{\tau_w} = \frac{r}{R} \qquad (3.18)$$

The volumetric flow rate is given by

$$Q = \int_0^R 2\pi r V_z \, \mathrm{d}r \tag{3.19}$$

Integration by parts leads to:

$$Q = \left[\pi r^2 V_z\right]_0^R + \int_0^R \pi r^2 \left(-\frac{\mathrm{d}V_z}{\mathrm{d}r}\right) \mathrm{d}r$$

For the no-slip boundary condition at the wall, the first term on the right hand side is identically zero and therefore:

$$Q = \int_0^R \pi r^2 \left(-\frac{\mathrm{d}V_z}{\mathrm{d}r}\right) \mathrm{d}r \tag{3.20}$$

$$V_z = 0 \quad \text{at} \quad r = R$$

Now changing the variable of integration from r to τ_{rz} using equation (3.18):

$$r = R\left(\frac{\tau_{rz}}{\tau_w}\right) \quad ; \quad \mathrm{d}r = R\frac{\mathrm{d}\tau_{rz}}{\tau_w}$$

When $r = 0$, $\tau_{rz} = 0$ and at the walls of the pipe when $r = R$, $\tau_{+z} = \tau_w$. Substitution in equation (3.20) gives:

$$Q = \frac{\pi R^3}{\tau_w^3} \int_0^{\tau_w} \tau_{rz}^2 \mathrm{f}(\tau_{rz}) \, \mathrm{d}\tau_{rz} \tag{3.21}$$

The velocity gradient (or the shear rate) term $(-\mathrm{d}V_z/\mathrm{d}r)$ has been replaced by a function of the corresponding shear stress via equation (1.10). The form of the function will therefore depend on the viscosity model chosen to describe the rheology of the fluid. Equation (3.21) can be used in two ways:

(i) to determine general non-Newtonian characteristics of a time-independent fluid, as demonstrated in Chapter 2 and in Section 3.2.5, or

(ii) to be integrated directly for a specific fluid model to obtain volumetric flow rate-pressure drop relationship. This is demonstrated for the flow of a power-law fluid, for which the shear rate is given by equation (3.3):

$$-\frac{\mathrm{d}V_z}{\mathrm{d}r} = \mathrm{f}(\tau_{rz}) = \left(\frac{\tau_{rz}}{m}\right)^{1/n} \tag{3.22}$$

Substitution of equation (3.22) into equation (3.21) followed by integration and re-arrangement gives:

$$V = \frac{Q}{\pi R^2} = \left(\frac{n}{3n+1}\right)\left(\frac{\tau_w}{m}\right)^{1/n} R \tag{3.23}$$

Substituting for $\tau_w = (R/2)(-\Delta p/L)$ in equation (3.23) gives:

$$V = \left(\frac{n}{3n+1}\right)\left(\frac{(-\Delta p)R}{2mL}\right)^{1/n} R \tag{3.23a}$$

which is identical to equation (3.6).

A similar analysis for a Bingham plastic fluid will lead to the same expression for Q as equation (3.13). Thus, these are alternative methods of obtaining flow rate-pressure gradient relation for any specific model to describe the fluid rheology. The scheme given above provides a quicker method of obtaining the relation between pressure gradient and flow rate, but has the disadvantage that it does not provide a means of obtaining the velocity profile.

Example 3.3

The shear-dependent viscosity of a commercial grade of polypropylene at 403 K can satisfactorily be described using the three constant Ellis fluid model (equation 1.15), with the values of the constants: $\mu_0 = 1.25 \times 10^4$ Pa·s, $\tau_{1/2} = 6900$ Pa and $\alpha = 2.80$. Estimate the pressure drop required to maintain a volumetric flow rate of 4 cm^3/s through a 50 mm diameter and 20 m long pipe. Assume the flow to be laminar.

Solution

Since we need the $Q - (-\Delta p)$ relation to solve this problem, such a relationship will be first derived using the generalised approach outlined in Section 3.2.4. For laminar flow in circular pipes, the Ellis fluid model is given as:

$$\tau_{rz} = \frac{\mu_0(-\mathrm{d}V_z/\mathrm{d}r)}{1 + (\tau_{rz}/\tau_{1/2})^{\alpha-1}} \tag{1.15}$$

$$\text{or} \quad -\frac{\mathrm{d}V_z}{\mathrm{d}r} = \mathrm{f}(\tau_{rz}) = \frac{1}{\mu_0}\left(\tau_{rz} + \frac{\tau_{rz}^{\alpha}}{\tau_{1/2}^{\alpha-1}}\right) \tag{3.24}$$

Substituting equation (3.24) into equation (3.21) followed by integration and rearrangement gives:

$$Q = \frac{\pi R^3 \tau_w}{4\mu_0}\left[1 + \left(\frac{4}{\alpha+3}\right)\left(\frac{\tau_w}{\tau_{1/2}}\right)^{\alpha-1}\right]$$

Note that in the limit of $\tau_{1/2} \to \infty$, i.e. for Newtonian fluid behaviour, this equation reduces to the Hagen–Poiseuille equation.

Now substituting the numerical values:

$$4 \times 10^{-6} = \frac{3.14 \times (0.025)^3 \tau_w}{4 \times 1.25 \times 10^4}\left[1 + \left(\frac{4}{2.8+3}\right)\left(\frac{\tau_w}{6900}\right)^{2.8-1}\right]$$

$$\text{or} \quad 4074.37 = \tau_w(1 + 8.4 \times 10^{-8}\tau_w^{1.8})$$

A trial and error procedure gives $\tau_w = 3412\,\text{Pa}$

$$\therefore \; -\Delta p = \frac{4\tau_w L}{D} = \frac{4 \times 3412 \times 20}{0.05} = 5.46 \times 10^6\,\text{Pa}$$

i.e. the pressure drop across the pipe will be 5.46 MPa.

3.2.5 Generalised Reynolds number for the flow of time-independent fluids

It is useful to define an appropriate Reynolds number which will result in a unique friction factor-Reynolds number curve for all time-independent fluids in laminar flow in circular pipes. Metzner and Reed [1955] outlined a generalised approach obviating this difficulty. The starting point is equation 3.21:

$$Q = \frac{\pi R^3}{\tau_w^3} \int_0^{\tau_w} \tau_{rz}^2 \mathrm{f}(\tau_{rz})\,\mathrm{d}\tau_{rz} \tag{3.21}$$

Equation (3.21) embodies a definite integral, the value of which depends only on the values of the integral function at the limits, and not on the nature of the continuous function that is integrated. For this reason it is necessary to evaluate only the wall shear stress τ_w and the associated velocity gradient at the wall $(-\mathrm{d}V_z/\mathrm{d}r)$ at $r = R$ or $\mathrm{f}(\tau_w)$. This is accomplished by the use of the Leibnitz rule which allows a differential of an integral of the form $(\mathrm{d}/\mathrm{d}s')\{\int_0^{s'} s^2 \mathrm{f}(s)\,\mathrm{d}s\}$ to be written as $(s')^2\mathrm{f}(s')$ where s is a dummy variable of integration (τ_{rz} here) and s' is identified as τ_w. First multiplying both sides of equation (3.21) by τ_w^3 and then differentiating with respect to τ_w gives:

$$\frac{\mathrm{d}}{\mathrm{d}\tau_w}\left\{\tau_w^3\left(\frac{Q}{\pi R^3}\right)\right\} = \frac{\mathrm{d}}{\mathrm{d}\tau_w}\int_0^{\tau_w} \tau_{rz}^2 \mathrm{f}(\tau_{rz})\,\mathrm{d}\tau_{rz}$$

Applying the Leibnitz rule to the integral on the right-hand side gives:

$$3\tau_w^2\left(\frac{Q}{\pi R^3}\right) + \tau_w^3 \frac{\mathrm{d}}{\mathrm{d}\tau_w}\left(\frac{Q}{\pi R^3}\right) = \tau_w^2 \mathrm{f}(\tau_w)$$

Introducing a factor of 4 on both sides and further rearrangement of the terms on the left-hand side gives:

$$\mathrm{f}(\tau_w) = \left(-\frac{\mathrm{d}V_z}{\mathrm{d}r}\right)_{\text{wall}} = \frac{4Q}{\pi R^3}\left[\frac{3}{4} + \frac{1}{4} \cdot \frac{\dfrac{\mathrm{d}(4Q/\pi R^3)}{(4Q/\pi R^3)}}{\dfrac{\mathrm{d}\tau_w}{\tau_w}}\right] \tag{3.25a}$$

or in terms of average velocity V and pipe diameter D,

$$\left(-\frac{\mathrm{d}V_z}{\mathrm{d}r}\right)_{\text{wall}} = \frac{8V}{D}\left\{\frac{3}{4} + \frac{1}{4}\frac{\mathrm{d}\log(8V/D)}{\mathrm{d}\log\tau_w}\right\} \tag{3.25b}$$

Here, $(8V/D)$ is the wall shear rate for a Newtonian fluid and is referred to as the nominal shear rate for a non-Newtonian fluid which is identical to equation (2.5) in Chapter 2. Alternatively, writing it in terms of the slope of $\log\tau_w - \log(8V/D)$ plot's,

$$\dot{\gamma}_w = \left(-\frac{\mathrm{d}V_z}{\mathrm{d}r}\right)_{\text{wall}} = \left(\frac{8V}{D}\right)\left(\frac{3n'+1}{4n'}\right) \tag{3.25c}$$

where $n' = (\mathrm{d}\log\tau_w/\mathrm{d}\log(8V/D))$ which is not necessarily constant at all shear rates. Equation (3.25c) is identical to equation (2.6) in Chapter 2.

Thus, the index n' is the slope of the log–log plots of the wall shear stress τ_w versus $(8V/D)$ in the laminar region (the limiting condition for laminar flow is discussed in Section 3.3). Plots of τ_w versus $(8V/D)$ thus describe the flow behaviour of time-independent non-Newtonian fluids and may be used directly for scale-up or process design calculations.

Over the range of shear rates over which n' is approximately constant, one may write a power-law type equation for this segment as

$$\tau_w = \frac{D}{4}\left(\frac{-\Delta p}{L}\right) = m'\left(\frac{8V}{D}\right)^{n'} \tag{3.26}$$

Substituting for τ_w in terms of the friction factor, f, $(= \tau_w/(1/2)\rho V^2)$, equation (3.26) becomes:

$$f = \frac{2}{\rho V^2}m'\left(\frac{8V}{D}\right)^{n'} \tag{3.27}$$

Now a Reynolds number may be defined so that in the laminar flow regime, it is related to f in the same way as is for Newtonian fluids, i.e.

$$f = \frac{16}{\mathrm{Re}_{MR}} \tag{3.28a}$$

from which

$$\mathrm{Re}_{MR} = \frac{\rho V^{2-n'}D^{n'}}{8^{n'-1}m'} \tag{3.28b}$$

Since Metzner and Reed [1955] seemingly were the first to propose this definition of the generalised Reynolds number, and hence the subscripts 'MR' in Re_{MR}.

It should be realised that by defining the Reynolds number in this way, the same friction factor chart can be used for Newtonian and time-independent non-Newtonian fluids in the laminar region. In effect, we are writing,

$$\text{Re}_{MR} = \frac{\rho V D}{\mu_{\text{eff}}} \tag{3.29}$$

Thus, the flow curve provides the value of the effective viscosity μ_{eff} where $\mu_{\text{eff}} = m'(8V/D)^{n'-1}$. It should be noted that the terms apparent and effective viscosity have been used to relate the behaviour of a non-Newtonian fluid to an equivalent property of a hypothetical Newtonian fluid. The apparent viscosity is the point value of the ratio of the shear stress to the shear rate. The effective viscosity is linked to the macroscopic behaviour $(Q - (-\Delta p))$ characteristics, for instance) and is equal to the Newtonian viscosity which would give the same relationship. It will be seen in Chapter 8 that this approach has also been quite successful in providing a reasonable basis for correlating much of the literature data on power consumption for the mixing of time-independent non-Newtonian fluids. The utility of this approach for reconciling the friction factor data for all time-independent fluids including shear-thinning and viscoplastic fluids, has been demonstrated by Metzner and Reed [1955] and subsequently by numerous other workers. Indeed, by writing a force balance on an element of fluid flowing in a circular pipe it can readily be shown that equation (3.28a) is also applicable for visco-elastic fluids. Figure 3.5 confirms this expectation for the flow of highly shear-thinning inelastic and visco-elastic polymer solutions in the range $0.28 \le n' \le 0.92$. [Chhabra *et al.*, 1984]. Griskey and Green [1971] have shown that the same approach may be adopted for the flow of shear-thickening materials, in the range $1.15 \le n' \le 2.50$. Experimental evidence suggests that stable laminar flow prevails for time-independent non-Newtonian fluids for Re_{MR} up to about 2000–2500; the transition from laminar to turbulent flow as well as the friction factor – Reynolds number characteristics beyond the laminar region are discussed in detail in the next section.

Before concluding this section, it is useful to link the apparent power-law index n' and consistency coefficient m' (equation 3.26) to the true power-law constants n and m, and to the Bingham plastic model constants τ_0^B and μ_B. This is accomplished by noting that $\tau_w = (D/4)(-\Delta p/L)$ always gives the wall shear stress and the corresponding value of the wall shear rate $\dot{\gamma}_w (= \mathrm{d}V_z/\mathrm{d}r)_w$ can be evaluated using the expressions for velocity distribution in a pipe presented in Sections 3.2.1 and 3.2.2.

For the laminar flow of a power-law fluid in a pipe, the velocity distribution over the pipe cross-section is given by equation (3.7a):

$$\frac{V_z}{V} = \left(\frac{3n+1}{n+1}\right)\left\{1 - \left(\frac{r}{R}\right)^{(n+1)/n}\right\} \tag{3.7a}$$

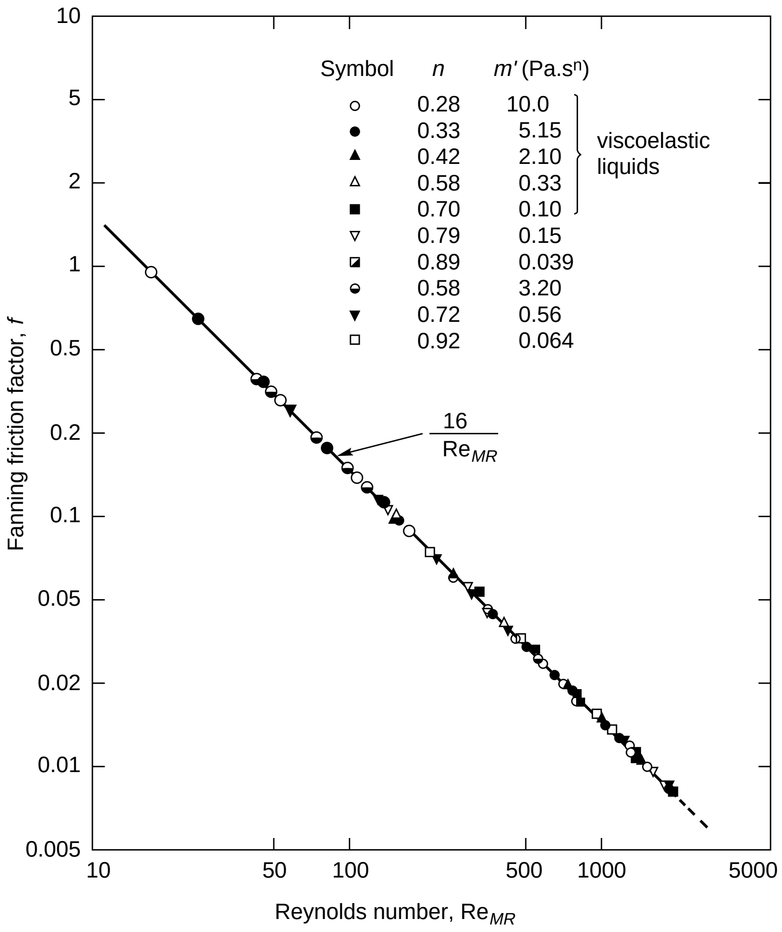

Figure 3.5 *Typical friction factor data for laminar flow of polymer solutions [Chhabra et al., 1984]*

Differentiating with respect to r and substituting $r = R$ gives the expression for the velocity gradient (or shear rate) at the wall as:

$$-\frac{\mathrm{d}V_z}{\mathrm{d}r} = \frac{3n+1}{n}\left(\frac{r}{R}\right)^{1/n}\frac{V}{R}$$

and:
$$\left(-\frac{\mathrm{d}V_z}{\mathrm{d}r}\right)_{r=R} = \dot{\gamma}_w = \left(\frac{3n+1}{n}\right)\left(\frac{V}{R}\right) = \left(\frac{3n+1}{4n}\right)\left(\frac{8V}{D}\right)$$

The corresponding value of the shear stress at the wall τ_w for a power-law fluid is obtained by substituting this value of $\dot{\gamma}_w$ in equation (3.3):

$$\tau_w = \tau_{rz}|_{r=R} = m\left(-\frac{\mathrm{d}V_z}{\mathrm{d}r}\right)^n_{r=R}$$

$$= m\left\{\left(\frac{3n+1}{4n}\right)\left(\frac{8V}{D}\right)\right\}^n$$

which is identical to equation (2.3) presented in Chapter 2. Now comparing this equation with equation (3.26) for a power-law fluid gives:

$$n' = n; \quad m' = m\left(\frac{3n+1}{4n}\right)^n \tag{3.30a}$$

In this case therefore n' and m' are both constant and independent of shear rate. Similarly, it can be shown that the Bingham model parameters τ_0^B and μ_B are related to m' and n' as [Skelland, 1967]:

$$n' = \frac{1 - \frac{4}{3}\phi + \frac{\phi^4}{3}}{1 - \phi^4} \tag{3.30b}$$

and

$$m' = \tau_w \left[\frac{\mu_B}{\tau_w\left(1 - \frac{4}{3}\phi + \frac{\phi^4}{3}\right)}\right]^{n'} \tag{3.30c}$$

where $\phi = (\tau_0^B/\tau_w)$.

It should be noted that in this case, m' and n' are not constant and depend on the value of the wall shear stress τ_w.