3.3. Criteria for transition from laminar to turbulent flow

For all fluids, the nature of the flow is governed by the relative importance of the viscous and the inertial forces. For Newtonian fluids, the balance between these forces is characterised by the value of the Reynolds number. The generally accepted value of the Reynolds number above which stable laminar flow no longer occurs is 2100 for Newtonian fluids. For time-independent fluids, the critical value of the Reynolds number depends upon the type and the degree of non-Newtonian behaviour. For power-law fluids ($n = n'$), the criterion of ,Ryan and Johnson [1959] can be used

$$\mathrm{Re}_{MR} = \frac{6464n}{(3n+1)^2}(2+n)^{(2+n)/(1+n)} \tag{3.31}$$

While for Newtonian fluids equation (3.31) predicts the critical Reynolds number of 2100, the corresponding limiting values increase with decreasing values of the power-law index, reaching a maximum of about 2400 at $n = 0.4$ and then dropping to 1600 at $n = 0.1$. The latter behaviour is not in line with the experimental results of Dodge and Metzner [1959] who observed laminar flow conditions up to $\text{Re}_{MR} \sim 3100$ for a fluid with $n' = 0.38$. Despite the complex dependence of the limiting Reynolds number on the flow behaviour index embodied in equation (3.31) and the conflicting experimental evidence, it is probably an acceptable approximation to assume that the laminar flow conditions cease to prevail at Reynolds numbers above ca. 2000–2500 and, for the purposes of process calculations, the widely accepted figure of 2100 can be used for time-independent fluids characterised in terms of n'. It is appropriate to add here that though the friction factor for visco-elastic fluids in the laminar regime is given by equation (3.28a), the limited experimental results available suggest much higher values for the critical Reynolds number. For instance, Metzner and Park [1964] reported that their friction factor data for visco-elastic polymer solutions were consistent with equation (3.28a) up to about $\text{Re}_{MR} = 10\,000$. However, it is not yet possible to put forward a quantitative criterion for calculating the limiting value of Re_{MR} for visco-elastic fluids.

Several other criteria, depending upon the use of a specific fluid model, are also available in the literature [Hanks, 1963; Govier and Aziz, 1982; Wilson, 1996; Malin, 1997]. For instance, Hanks [1963] proposed the following criterion for Bingham plastic fluids:

$$(\text{Re}_B)_c = \frac{\rho V D}{\mu_B} = \frac{1 - \dfrac{4}{3}\phi_c + \dfrac{\phi_c^4}{3}}{8\phi_c} He \tag{3.32a}$$

where the ratio, $\phi_c = (\tau_0^B/\tau_{w_c})$, is given by:

$$\frac{\phi_c}{(1-\phi_c)^3} = \frac{He}{16\,800} \tag{3.32b}$$

The Hedström number, He, is defined as:

$$He = \frac{\rho D^2 \tau_0^B}{\mu_B^2} = \text{Re}_B \times \text{Bi} \tag{3.33}$$

where $\text{Bi} = (D\tau_0^B/\mu_B V)$ is the Bingham number. For a given pipe size (D) and Bingham plastic fluid behaviour (ρ, μ_B, τ_0^B), the Hedström number will be known and the value of ϕ_c can be obtained from equation (3.32b) which, in turn, facilitates the calculation of $(\text{Re}_B)_c$ using equation (3.32a), as illustrated in example 3.4. More recent numerical calculations [Malin, 1997] lend further support to the validity of equations (3.32a,b).

Both Wilson [1996] and Slatter [1996] have also re-evaluated the available criteria for the laminar–turbulent transition, with particular reference to the flow of pseudoplastic and yield-pseudoplastic mineral slurries in circular pipes. Wilson [1996] has argued that the larger dissipative micro-eddies present in the wall region result in thicker viscous sub-layers in non-Newtonian fluids which, in turn, produce greater mean velocity, giving a friction factor lower than that for Newtonian fluids, for the same value of the pressure drop across the pipe. For power-law fluids, he was able to link the non-Newtonian apparent viscosity to the viscosity of a hypothetical Newtonian fluid simply through a function of n, the power-law flow behaviour index, such that the same $Q - (-\Delta p)$ relationship applies to both fluids. This, in turn, yields the criterion for laminar–turbulent transition in terms of the critical value of the friction factor as a function of n (power-law index) alone. Note that in this approach, the estimated value of the effective viscosity will naturally depend upon the type of fluid and pipe diameter, D. Similarly, Slatter [1996] has put forward a criterion in terms of a new Reynolds number for the flow of Herschel–Bulkley model fluids (equation (1.17)) to delineate the laminar–turbulent transition condition. His argument hinges on the fact that the inertial and viscous forces in the fluid are determined solely by that part of the fluid which is undergoing deformation (shearing), and hence he excluded that part of the volumetric flow rate attributable to the unsheared plug of material present in the middle of the pipe. These considerations lead to the following definition of the modified Reynolds number:

$$\mathrm{Re}_{\mathrm{mod}} = \frac{8\rho V_{\mathrm{ann}}^2}{\tau_0^H + m\left(\dfrac{8V_{\mathrm{ann}}}{D_{\mathrm{shear}}}\right)^n} \tag{3.34}$$

where $$V_{\mathrm{ann}} = \frac{Q - Q_{\mathrm{plug}}}{\pi(R^2 - R_p^2)}, \quad \text{and } D_{\mathrm{shear}} = 2(R - R_p)$$

Laminar flow conditions cease to exist at $\mathrm{Re}_{\mathrm{mod}} = 2100$. The calculation of the critical velocity corresponding to $\mathrm{Re}_{\mathrm{mod}} = 2100$ requires an iterative procedure. For known rheology (ρ, m, n, τ_0^H) and pipe diameter (D), a value of the wall shear stress is assumed which, in turn, allows the calculation of R_p, from equation (3.9), and Q and Q_p from equations (3.14b) and (3.14a) respectively. Thus, all quanties are then known and the value of $\mathrm{Re}_{\mathrm{mod}}$ can be calculated. The procedure is terminated when the value of τ_w has been found which makes $\mathrm{Re}_{\mathrm{mod}} = 2100$, as illustrated in example 3.4 for the special case of $n = 1$, i.e., for the Bingham plastic model, and in example 3.5 for a Herschel–Bulkley fluid. Detailed comparisons between the predictions of equation (3.34) and experimental data reveal an improvement in the predictions, though the values of the critical velocity obtained using the criterion $\mathrm{Re}_{MR} = 2100$ are only 20–25% lower than those predicted by equation (3.34). Furthermore, the two criteria coincide for power-law model fluids. Subsequently, it has also been shown that while the laminar–turbulent transition in small diameter tubes is virtually unaffected by the value of the yield stress, both the flow behaviour index (n) and the yield stress play increasingly greater roles in determining the transition point with increasing pipe diameter. Finally, the scant results obtained with a kaolin slurry and a CMC solution seem to suggest that the laminar–turbulent transition is not influenced by the pipe roughness [Slatter, 1996, 1997].

Example 3.4

The rheological behaviour of a coal slurry (1160 kg/m^3) can be approximated by the Bingham plastic model with $\tau_0^B = 0.5$ Pa and $\mu_B = 14$ mPa·s. It is to be pumped through a 400 mm diameter pipe at the rate of 188 kg/s. Ascertain the nature of the flow by calculating the maximum permissible velocity for laminar flow conditions. Contrast the predictions of equations (3.33) and (3.34).

Solution

Here, the Hedström number, $\text{He} = \dfrac{\rho D^2 \tau_0^B}{\mu_B^2}$

$$= \frac{1160 \times 0.4^2 \times 0.5}{(14 \times 10^{-3})^2}$$

i.e. $He = 4.73 \times 10^5$ which when substituted in equation (3.32b) yields,

$$\frac{\phi_c}{(1-\phi_c)^3} = \frac{4.73 \times 10^5}{16\,800} = 28.15$$

A trial and error procedure gives $\phi_c = 0.707$. Now substituting for He and ϕ_c in equation (3.32a):

$$(\text{Re}_B)_c = \frac{1 - \dfrac{4}{3} \times 0.707 + \dfrac{(0.707)^4}{3}}{8 \times 0.707} \times 4.73 \times 10^5$$

$$\text{or} \quad (\text{Re}_B)_c = \frac{\rho V_c D}{\mu_B} = 11\,760$$

and the maximum permissible velocity, V_c therefore is,

$$V_c = \frac{11\,760 \times 14 \times 10^{-3}}{1160 \times 0.4} = 0.354 \text{ m/s.}$$

The actual velocity in the pipe is

$$\left(\frac{188}{1160}\right)\left(\frac{4}{\pi \times 0.4^2}\right) = 1.29 \text{ m/s}$$

Thus, the flow in the pipe is not streamline.

Alternatively, one can use equation (3.34) to estimate the maximum permissible velocity for streamline flow in the pipe. In this example, $n = 1$, $m = 0.014$ Pa and $\tau_0^H = 0.5$ Pa. As mentioned previously the use of equation (3.34) requires an iterative procedure, and to initiate this method let us assume a value of $\tau_w = 0.6$ Pa.

$$\therefore\ \phi = \frac{\tau_0^H}{\tau_w} = \frac{0.5}{0.6} = \frac{R_p}{R},\ \text{ i.e. } R_p = 0.166\,\text{m and}$$

$\phi = 0.833$. Now using equation (3.14b) for $n = 1$:

$$Q = \pi \times 0.2^3 \times 1 \left(\frac{0.6}{0.014}\right.\quad (1 - 0.833)^2$$

$$\times \quad \frac{(1-0.833)^2}{4} + \frac{2 \times 0.833(1-0.833)}{3} + \frac{0.833^2}{2}$$

$$= 0.0134\,\text{m}^3/\text{s}$$

The plug velocity, V_p, is calculated from equation (3.14a) by setting $r/R = R_p/R = \phi = 0.833$, i.e.

$$V_p = \frac{1 \times 0.2}{(1+1)}\left(\frac{0.6}{0.014}\right.\quad (1-0.833)^2 = 0.1195\,\text{m/s}$$

$$Q_p = V_p \pi R_p^2 = 0.1195 \times \pi \times 0.166^2 = 0.01035\,\text{m}^3/\text{s}$$

$$\therefore\quad Q_{\text{ann}} = Q - Q_p = 0.0134 - 0.01035 = 0.00305\,\text{m}^3/\text{s}$$

$$V_{\text{ann}} = \frac{Q_{\text{ann}}}{\pi(R^2 - R_p^2)} = \frac{0.00305}{\pi(0.2^2 - 0.166^2)} = 0.078\,\text{m/s}$$

$$D_{\text{shear}} = 2(R - R_p) = 2(0.2 - 0.166) = 0.068\,\text{m}$$

$$\therefore\quad \text{Re}_{\text{mod}} = \frac{8\rho V_{\text{ann}}^2}{\tau_0^H + m\left(\frac{8V_{\text{ann}}}{D_{\text{shear}}}\right.^{n}} = \frac{8 \times 1160 \times 0.078^2}{0.5 + 0.014\left(\frac{8 \times 0.078}{0.068}\right.} = 90$$

which is too small for the flow to be turbulent. Thus, this procedure must be repeated for other values of τ_w to make $\text{Re}_{\text{mod}} = 2100$. A summary of calculations is presented in the table below.

τ_w (Pa)	Q (m^3/s)	Q_p (m^3/s)	Re_{mod}
0.6	0.0134	0.01035	90
0.70	0.0436	0.0268	890
0.73	0.0524	0.0305	1263
0.80	0.0781	0.0395	2700
0.77	0.0666	0.0358	1994
0.78	0.0706	0.0370	2233
0.775	0.0688	0.0365	2124

The last entry is sufficiently close to $\text{Re}_{\text{mod}} = 2100$, and laminar flow will cease to exist at $\tau_w \geq 0.775$ Pa. Also, note that the use of equations (3.14a) and (3.14b) beyond this value of wall shear stress is incorrect.

$$\therefore \text{ maximum permissible velocity} = \frac{0.0688}{\frac{\pi}{4}(0.4)^2} = 0.55 \text{ m/s.}$$

This value is some 40% higher than the previously calculated value of 0.35 m/s. However, even on this count, the flow will be turbulent at the given velocity of 1.29 m/s.

Example 3.5

Determine the critical velocity for the upper limit of laminar flow for a slurry with the following properties, flowing in a 150 mm diameter pipe.

$$\rho = 1150 \text{ kg/m}^3; \quad \tau_0^H = 6 \text{ Pa}; \quad m = 0.3 \text{ Pa·s}^n \quad \text{and} \quad n = 0.4$$

Solution

As in example 3.4, one needs to assume a value for τ_w, and then to calculate all other quantities using equations (3.14a) and (3.14b) which in turn allow the calculation of Re_{mod} using equation (3.34). A summary of the calculations is presented here in a tabular form.

τ_w (Pa)	Q (m^3/s)	Q_p (m^3/s)	Re_{mod}
6.4	4.72×10^{-5}	4.27×10^{-5}	6.5×10^{-3}
7.4	3.097×10^{-3}	2.216×10^{-3}	26.6
8.4	0.01723	0.01	778
9.3	0.046	0.0224	5257
8.82	0.0287	0.0153	2100

Thus, the laminar–turbulent transition for this slurry in a 150 mm diameter pipe occurs when the wall shear stress is 8.82 Pa and the volumetric flowrate is 0.0287 m^3/s.

$$\therefore \text{ mean velocity at this point} = \frac{Q}{(\pi/4)D^2} = \frac{0.0287}{(\pi/4)(0.15)^2} = 1.62 \text{ m/s}$$

Hence, streamline flow will occur for this slurry in a 150 mm diameter pipe at velocities up to a value of 1.62 m/s.