L'Hopital's rule

Suppose that we have one of the following cases,

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{0}{0} \qquad \text{or} \quad \lim_{x\to c}\frac{f(x)}{g(x)}=\frac{\mp\infty}{\mp\infty}$$

where c can be any real number, infinity or negative infinity. In these cases we have,

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

Example: Evaluate the limits

1. $\displaystyle\lim_{x\to 0}\frac{e^x-x-1}{x^2}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{e^x-x-1}{x^2}=\lim_{x\to 0}\frac{e^x-1}{2x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{e^x}{2}=\frac{1}{2}$$

2. $\displaystyle\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x-\sin x}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x-\sin x}=\lim_{x\to 0}\frac{2\cos x-2\cos 2x}{1-\cos x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-2\sin x+4\sin 2x}{\sin x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-2\cos x+8\cos 2x}{\cos x}=\frac{-2+8}{1}=6$$

3. $\displaystyle\lim_{x\to\frac{1}{2}}\frac{(1-4x^2)\sin^{-1}x}{\cos(\pi x)}=\lim_{x\to\frac{1}{2}}\sin^{-1}x\times\lim_{x\to\frac{1}{2}}\frac{1-4x^2}{\cos(\pi x)}=\sin^{-1}\left(\frac{1}{2}\right)\lim_{x\to\frac{1}{2}}\frac{-8x}{-\pi\sin(\pi x)}$

$$=\frac{\pi}{6}\times\frac{4}{\pi\sin\left(\frac{\pi}{2}\right)}=\frac{\pi}{6}\times\frac{4}{\pi}=\frac{2}{3}$$

4. $\displaystyle\lim_{x\to\infty}\frac{5+\ln 2x}{x^2+4}=\frac{\infty}{\infty}$

$$\lim_{x\to\infty}\frac{5+\ln 2x}{x^2+4}=\lim_{x\to\infty}\frac{\frac{2}{2x}}{2x}=\lim_{x\to\infty}\frac{1}{2x^2}=\frac{1}{\infty}=0$$

5. $\lim_{x\to\infty} \frac{(\ln x)^2}{e^{2x}} = \frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{(\ln x)^2}{e^{2x}} = \lim_{x\to\infty} \frac{(\ln x)^2}{(e^x)^2} = \left(\lim_{x\to\infty} \frac{\ln x}{e^x}\right)^2 = \left(\lim_{x\to\infty} \frac{\frac{1}{x}}{e^x}\right)^2 = \left(\frac{0}{\infty}\right)^2 = 0$$

6. $\lim_{x\to 0} \frac{x^4 e^{3x}}{(e^{2x}-1)^2} = \frac{0}{0}$

$$\lim_{x\to 0} \frac{x^4 e^{3x}}{(e^{2x}-1)^2} = \lim_{x\to 0} \frac{(x^2)^2}{(e^{2x}-1)^2} \times \lim_{x\to 0} e^{3x} = \left(\lim_{x\to 0} \frac{x^2}{(e^{2x}-1)}\right)^2 \times 1$$

$$= \left(\lim_{x\to 0} \frac{2x}{2e^{2x}}\right)^2 = \left(\lim_{x\to 0} \frac{2}{4e^{2x}}\right)^2 = \left(\frac{2}{4}\right)^2 = \frac{1}{4}$$

Exercises

Evaluate the limits

1. $\lim_{x\to 0} \frac{\ln(1+x)-x}{\cos x - 1}$

2. $\lim_{x\to 0} \frac{\cos 4x - \cos 2x}{x^2}$

3. $\lim_{x\to\infty} \frac{7-\ln(2x+3)}{\ln(5x-2)}$

4. $\lim_{x\to -1} \frac{1+\cos\pi x}{x^2+2x+1}$

5. $\lim_{x\to 4} \frac{\sin^2(\pi x)}{e^{x-4}-x+3}$

6. $\lim_{x\to 0} \frac{x^3}{x-\sin x}$

7. $\lim_{x\to 0} \frac{e^x-1-x}{x\sin x}$

8. $\lim_{x\to\frac{\pi}{4}} \frac{\sin 4x}{x(\cos x - \sin x)}$