The Method of Distribution Functions

We will illustrate the method of distribution functions with a simple univariate example. If Y has probability density function $f(y)$ and if U is some function of Y, then we can find $F_U(u) = P(U \leq u)$ directly by integrating $f(y)$ over the region for which $U \leq u$. The probability density function for U is found by differentiating $F_U(u)$. The following example illustrates the method.

Summary of the Distribution Function Method

Let U be a function of the random variables $Y_1, Y_2, \ldots, Y_n$.

1. Find the region $U = u$ in the $(y_1, y_2, \ldots, y_n)$ space.
2. Find the region $U \leq u$.
3. Find $F_U(u) = P(U \leq u)$ by integrating $f(y_1, y_2, \ldots, y_n)$ over the region $U \leq u$.
4. Find the density function $f_U(u)$ by differentiating $F_U(u)$. Thus, $f_U(u) = dF_U(u)/du$.

Discrete Random variable

First, if X is a discrete random variable with mass points $x_1, x_2, \ldots,$ then the distribution of $Y = g(X)$ is determined directly by the laws of probability. If X takes on the values $x_1, x_2, \ldots$ with probabilities $f_X(x_1), f_X(x_2), \ldots,$ then the possible values of Y are determined by substituting the successive values of X in $g(\cdot)$. It may be that several values of X give rise to the same value of Y. The probability that Y takes on a given value, say y_j, is

$$f_Y(y_j) = \sum_{\{i: g(x_i) = y_j\}} f_X(x_i).$$

EXAMPLE 15 Suppose X takes on the values 0, 1, 2, 3, 4, 5 with probabilities $f_X(0), f_X(1), f_X(2), f_X(3), f_X(4)$, and $f_X(5)$. If $Y = g(X) = (X - 2)^2$, note that Y can take on values 0, 1, 4, and 9; then $f_Y(0) = f_X(2)$, $f_Y(1) = f_X(1) + f_X(3)$, $f_Y(4) = f_X(0) + f_X(4)$, and $f_Y(9) = f_X(5)$. ////

EXAMPLE 1 Let there be only one given random variable, say X, which has a standard normal distribution. Suppose the distribution of $Y = g(X) = X^2$ is desired.

$$\begin{aligned} F_Y(y) &= P[Y \leq y] = P[X^2 \leq y] = P[-\sqrt{y} \leq X \leq \sqrt{y}] = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) \\ &= 2\int_0^{\sqrt{y}} \phi(u)\, du = 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du \\ &= \frac{2}{\sqrt{2\pi}} \int_0^y \frac{1}{2\sqrt{z}} e^{-\frac{1}{2}z}\, dz = \int_0^y \frac{1}{\Gamma(\frac{1}{2})} \frac{1}{\sqrt{2z}} e^{-\frac{1}{2}z}\, dz, \quad \text{for } y > 0, \end{aligned}$$

which can be recognized as the cumulative distribution function of a gamma distribution with parameters $r = \frac{1}{2}$ and $\lambda = \frac{1}{2}$. ////

EXAMPLE

A process for refining sugar yields up to 1 ton of pure sugar per day, but the actual amount produced, Y, is a random variable because of machine breakdowns and other slowdowns. Suppose that Y has density function given by

$$f(y) = \begin{cases} 2y, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Solution

The company is paid at the rate of \$300 per ton for the refined sugar, but it also has a fixed overhead cost of \$100 per day. Thus the daily profit, in hundreds of dollars, is $U = 3Y - 1$. Find the probability density function for U.

To employ the distribution function approach, we must find

$$F_U(u) = P(U \le u) = P(3Y - 1 \le u) = P\left(Y \le \frac{u+1}{3}\right).$$

If $u < -1$, then $(u+1)/3 < 0$ and, therefore, $F_U(u) = P(Y \le (u+1)/3) = 0$. Also, if $u > 2$, then $(u+1)/3 > 1$ and $F_U(u) = P(Y \le (u+1)/3) = 1$. However, if $-1 \le u \le 2$, the probability can be written as an integral of $f(y)$, and

$$P\left(Y \le \frac{u+1}{3}\right) = \int_{-\infty}^{(u+1)/3} f(y)dy = \int_0^{(u+1)/3} 2y\, dy = \left(\frac{u+1}{3}\right)^2.$$

(Notice that, as Y ranges from 0 to 1, U ranges from -1 to 2.) Thus, the distribution function of the random variable U is given by

$$F_U(u) = \begin{cases} 0, & u < -1, \\ \left(\frac{u+1}{3}\right)^2, & -1 \le u \le 2, \\ 1, & u > 2, \end{cases}$$

and the density function for U is

$$f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} (2/9)(u+1), & -1 \le u < 2, \\ 0, & \text{elsewhere.} \end{cases}$$ ■

Second, if X is a continuous random variable, then the cumulative distribution function of $Y = g(X)$ can be found by integrating $f_X(x)$ over the appropriate region; that is,

$$F_Y(y) = P[Y \leq y] = P[g(X) \leq y] = \int_{\{x:g(x)\leq y\}} f_X(x)\,dx. \tag{36}$$

This is just the cumulative-distribution-function technique.

EXAMPLE 16 Let X be a random variable with uniform distribution over the interval $(0, 1)$ and let $Y = g(X) = X^2$. The density of Y is desired. Now

$$F_Y(y) = P[Y \leq y] = P[X^2 \leq y] = \int_{\{x:x^2\leq y\}} f_X(x)\,dx = \int_0^{\sqrt{y}} dx = \sqrt{y}$$

for $0 < y < 1$; so

$$F_Y(y) = \sqrt{y}I_{(0,1)}(y) + I_{[1,\infty)}(y),$$

and therefore

$$f_Y(y) = \frac{1}{2}\frac{1}{\sqrt{y}} I_{(0,1)}(y).$$ ////

MOMENT-GENERATING-FUNCTION TECHNIQUE

The moment-generating function method for finding the probability distribution of a function of random variables $Y_1, Y_2, \ldots, Y_n$ is based on the following uniqueness theorem.

Let $m_X(t)$ and $m_Y(t)$ denote the moment-generating functions of random variables X and Y, respectively. If both moment-generating functions exist and $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.

EXAMPLE 6 Suppose X has a normal distribution with mean 0 and variance 1. Let $Y = X^2$, and find the distribution of Y.

$$m_Y(t) = \mathscr{E}[e^{tY}] = \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2(1-2t)}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{(1-2t)^{-\frac{1}{2}}}{(1-2t)^{-\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2(1-2t)}\, dx$$
$$= (1-2t)^{-\frac{1}{2}} = \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{\frac{1}{2}} \qquad \text{for} \quad t < \frac{1}{2},$$

which we recognize as the moment generating function of a gamma with parameters $r = \frac{1}{2}$ and $\lambda = \frac{1}{2}$. (It is also called a chi-square distribution with one degree of freedom. See Subsec. 4.3 of Chap. VI.) ////

EXAMPLE Let X_1 and X_2 be two independent standard normal random variables. Let $Y_1 = g_1(X_1, X_2) = X_1 + X_2$ and $Y_2 = g_2(X_1, X_2) = X_2 - X_1$. Find the joint distribution of Y_1 and Y_2.

$$\begin{aligned}
m_{Y_1, Y_2}(t_1, t_2) &= \mathscr{E}[e^{Y_1 t_1 + Y_2 t_2}] \\
&= \mathscr{E}[e^{(X_1 + X_2)t_1 + (X_2 - X_1)t_2}] \\
&= \mathscr{E}[e^{X_1(t_1 - t_2) + X_2(t_1 + t_2)}] \\
&= \mathscr{E}[e^{X_1(t_1 - t_2)}]\mathscr{E}[e^{X_2(t_1 + t_2)}] \\
&= m_{X_1}(t_1 - t_2) m_{X_2}(t_1 + t_2) \\
&= \exp \frac{(t_1 - t_2)^2}{2} \exp \frac{(t_1 + t_2)^2}{2} \\
&= \exp(t_1^2 + t_2^2) = \exp \frac{2t_1^2}{2} \exp \frac{2t_2^2}{2} \\
&= m_{Y_1}(t_1) m_{Y_2}(t_2).
\end{aligned}$$

We note that Y_1 and Y_2 are independent random variablesand each has a normal distribution with mean 0 :

and variance

EXAMPLE 8 Let X_1 and X_2 be two independent standard normal randon variables. Let $Y = (X_2 - X_1)^2/2$, and find the distribution of Y.

$$m_Y(t) = \mathscr{E}[\exp Yt] = \mathscr{E}\left[\exp \frac{(X_2 - X_1)^2}{2} t\right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left[\frac{(x_2 - x_1)^2}{2} t - \frac{x_1^2 + x_2^2}{2}\right] dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{-\frac{1}{2}[x_1^2(1-t) + 2x_1x_2t + x_2^2(1-t)]\right\} dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left[-\frac{1}{2} x_2^2(1-t)\right]$$

$$\times \left\{\int_{-\infty}^{\infty} \exp\left[-\frac{1-t}{2}\left(x_1^2 + \frac{2x_1x_2t}{1-t}\right)\right] dx_1\right\} dx_2$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x_2^2(1-t)}{2}\right] \exp \frac{x_2^2 t^2}{2(1-t)}$$

$$\times \frac{1}{\sqrt{1-t}} \left\{\int_{-\infty}^{\infty} \frac{\sqrt{1-t}}{\sqrt{2\pi}} \exp\left[-\frac{1-t}{2}\left(x_1 + \frac{x_2 t}{1-t}\right)^2\right] dx_1\right\} dx_2$$

$$= \frac{1}{\sqrt{1-t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(1 - t - \frac{t^2}{1-t}\right) x_2^2\right] dx_2$$

$$= \frac{1}{\sqrt{1-t}} \cdot \frac{\sqrt{1-t}}{\sqrt{1-2t}} \cdot \frac{\sqrt{1-2t}}{\sqrt{1-t}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \frac{1-2t}{1-t} x_2^2\right) dx_2$$

$$= (1-2t)^{-\frac{1}{2}} = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{\frac{1}{2}}, \text{ for } t < 1/2,$$

which is the moment generating function of a gamma distribution with parameters $r = \frac{1}{2}$ and $\lambda = \frac{1}{2}$; hence,

$$f_Y(y) = [\sqrt{\tfrac{1}{2}}/\Gamma(\tfrac{1}{2})] y^{-\frac{1}{2}} e^{-y/2} I_{(0,\infty)}(y). \qquad ////$$

EXAMPLE Suppose that Y is a normally distributed random variable with mean μ and variance σ^2. Show that

$$Z = \frac{Y - \mu}{\sigma}$$

has a *standard normal* distribution, a normal distribution with mean 0 and variance 1.

Solution We have seen in Example 4.16 that $Y - \mu$ has moment-generating function $e^{t^2\sigma^2/2}$. Hence,

$$m_Z(t) = E(e^{tZ}) = E[e^{(t/\sigma)(Y-\mu)}] = m_{(Y-\mu)}\left(\frac{t}{\sigma}\right) = e^{(t/\sigma)^2(\sigma^2/2)} = e^{t^2/2}.$$

On comparing $m_Z(t)$ with the moment-generating function of a normal random variable, we see that Z must be normally distributed with $E(Z) = 0$ and $V(Z) = 1$.

Distribution of Sums of Independent Random Variables

In this subsection we employ the moment-generating-function technique to find the distribution of the sum of independent random variables.

Theorem 9 If $X_1, \ldots, X_n$ are independent random variables and the moment generating function of each exists for all $-h < t < h$ for some $h > 0$, let $Y = \sum_1^n X_i$; then

$$m_Y(t) = \mathscr{E}[\exp \sum X_i t] = \prod_{i=1}^n m_{X_i}(t) \qquad \text{for} \quad -h < t < h.$$

PROOF

$$m_Y(t) = \mathscr{E}[\exp \sum X_i t] = \mathscr{E}\left[\prod_{i=1}^n e^{X_i t}\right]$$
$$= \prod_{i=1}^n \mathscr{E}[e^{X_i t}] = \prod_{i=1}^n m_{X_i}(t)$$

EXAMPLE 9 Suppose that $X_1, \ldots, X_n$ are independent Bernoulli random variables; that is, $P[X_i = 1] = p$, and $P[X_i = 0] = 1 - p$. Now

$$m_{X_i}(t) = pe^t + q.$$

So

$$m_{\sum X_i}(t) = \prod_{i=1}^n m_{X_i}(t) = (pe^t + q)^n,$$

EXAMPLE 10 Suppose that $X_1, \ldots, X_n$ are independent Poisson distributed random variables, X_i having parameter λ_i. Then

$$m_{X_i}(t) = \mathscr{E}[e^{tX_i}] = \exp \lambda_i(e^t - 1),$$

and hence

$$m_{\Sigma X_i}(t) = \prod_{i=1}^n m_{X_i}(t) = \prod_{i=1}^n \exp \lambda_i(e^t - 1) = \exp \sum \lambda_i(e^t - 1),$$

EXAMPLE 11 Assume that $X_1, \ldots, X_n$ are independent and identically distributed exponential random variables; then

$$m_{X_i}(t) = \frac{\lambda}{\lambda - t}.$$

So

$$m_{\sum X_i}(t) = \prod_{i=1}^{n} m_{X_i}(t) = \left(\frac{\lambda}{\lambda - t}\right)^n,$$

which is the moment generating function of a gamma distribution with parameters n and λ; hence,

$$m_{\sum a_i X_i}(t) = \prod_{i=1}^{n} m_{a_i X_i}(t) = \exp[(\sum a_i \mu_i)t + \tfrac{1}{2}(\sum a_i^2 \sigma_i^2)t^2],$$

which is the moment generating function of a normal random variable; so

$$\sum_1^n a_i X_i \sim N\left(\sum_1^n a_i \mu_i, \sum_1^n a_i^2 \sigma_i^2\right).$$

EXAMPLE 12 Assume that $X_1, \ldots, X_n$ are independent random variables and

$$X_i \sim N(\mu_i, \sigma_i^2);$$

then

$$a_i X_i \sim N(a_i \mu_i, a_i^2 \sigma_i^2),$$

and

$$m_{a_i X_i}(t) = \exp(a_i \mu_i t + \tfrac{1}{2} a_i^2 \sigma_i^2 t^2).$$

$$X \sim N(\mu_X, \sigma_X^2), \qquad Y \sim N(\mu_Y, \sigma_Y^2),$$

and X and Y are independent, then

$$X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2),$$

and

$$X - Y \sim N(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2).$$

If $X_1, \ldots, X_n$ are independent and identically distributed random variables distributed $N(\mu, \sigma^2)$, then

$$\bar{X}_n = \frac{1}{n} \sum X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right);$$

THE TRANSFORMATION $Y = g(X)$

A random variable X may be transformed by some function $g(\cdot)$ to define a new random variable Y. The density of Y, $f_Y(y)$, will be determined by the transformation $g(\cdot)$ together with the density $f_X(x)$ of X.

Theorem 11 Suppose X is a continuous random variable with probability density function $f_X(\cdot)$. Set $\mathfrak{X} = \{x : f_X(x) > 0\}$. Assume that:

(i) $y = g(x)$ defines a one-to-one transformation of $\mathfrak{X}$ onto $\mathfrak{Y}$.
(ii) The derivative of $x = g^{-1}(y)$ with respect to y is continuous and nonzero for $y \in \mathfrak{Y}$, where $g^{-1}(y)$ is the inverse function of $g(x)$; that is, $g^{-1}(y)$ is that x for which $g(x) = y$.

Then $Y = g(X)$ is a continuous random variable with density

$$f_Y(y) = \left| \frac{d}{dy} g^{-1}(y) \right| f_X(g^{-1}(y)) I_{\mathfrak{Y}}(y).$$

EXAMPLE Suppose X has a beta distribution. What is the distribution of $Y = -\log_e X$? $\mathfrak{X} = \{x : f_X(x) > 0\} = \{x : 0 < x < 1\}$. $y = g(x) = -\log_e x$ defines a one-to-one transformation of $\mathfrak{X}$ onto $\mathfrak{Y} = \{y: \ y > 0\}$. $x = g^{-1}(y) = e^{-y}$, so $(d/dy)g^{-1}(y) = -e^{-y}$, which is continuous and nonzero for $y \in \mathfrak{Y}$. By Theorem 11,

$$\begin{aligned} f_Y(y) &= \left| \frac{d}{dy} g^{-1}(y) \right| f_X(g^{-1}(y)) I_{\mathfrak{Y}}(y) \\ &= e^{-y} \frac{1}{\mathrm{B}(a, b)} (e^{-y})^{a-1} (1 - e^{-y})^{b-1} I_{(0, \infty)}(y) \\ &= \frac{1}{\mathrm{B}(a, b)} e^{-ay} (1 - e^{-y})^{b-1} I_{(0, \infty)}(y). \end{aligned}$$

In particular, if $b = 1$, then $\mathrm{B}(a, b) = 1/a$; so $f_Y(y) = ae^{-ay} I_{(0, \infty)}(y)$, an exponential distribution with parameter a. ////

Theorem Let X_1 and X_2 be jointly continuous random variables with density function $f_{X_1,X_2}(x_1, x_2)$. Set $\mathfrak{X} = \{(x_1, x_2): f_{X_1,X_2}(x_1, x_2) > 0\}$. Assume that:

(i) $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ defines a one-to-one transformation of $\mathfrak{X}$ onto $\mathfrak{Y}$.

(ii) The first partial derivatives of $x_1 = g_1^{-1}(y_1, y_2)$ and $x_2 = g_2^{-1}(y_1, y_2)$ are continuous over $\mathfrak{Y}$.

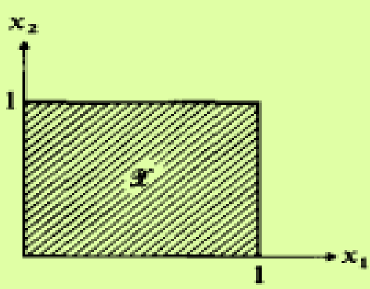
(iii) The Jacobian of the transformation is nonzero for $(y_1, y_2) \in \mathfrak{Y}$.

Then the joint density of $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ is given by

$$f_{Y_1,Y_2}(y_1, y_2) = |J| f_{X_1,X_2}(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)) I_{\mathfrak{Y}}(y_1, y_2).$$

EXAMPLE Suppose that X_1 and X_2 are independent random variabl each uniformly distributed over the interval (0, 1). Then $f_{X_1,X_2}(x_1, x_2)$ $I_{(0,1)}(x_1)I_{(0,1)}(x_2)$. $\mathfrak{X} = \{(x_1, x_2): 0 < x_1 < 1$ and $0 < x_2 < 1\}$. L $y_1 = g_1(x_1, x_2) = x_1 + x_2$ and $y_2 = g_2(x_1, x_2) = x_2 - x_1$; then x_1 $\frac{1}{2}(y_1 - y_2) = g_1^{-1}(y_1, y_2)$, and $x_2 = \frac{1}{2}(y_1 + y_2) = g_2^{-1}(y_1, y_2)$.

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$



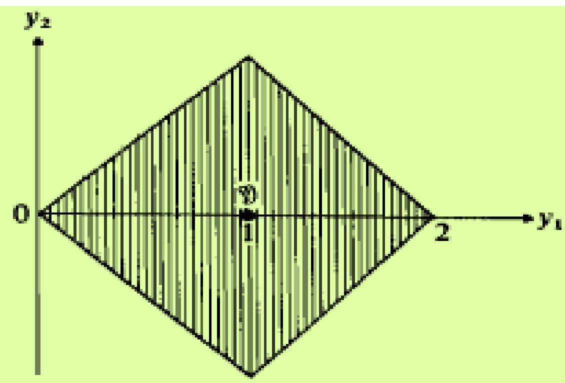


FIGURE 3

$\mathfrak{X}$ and $\mathfrak{Y}$ are sketched in Fig. 3. Note that the boundary $x_1 = 0$ of $\mathfrak{X}$ goes into the boundary $\frac{1}{2}(y_1 - y_2) = 0$ of $\mathfrak{Y}$, the boundary $x_2 = 0$ of $\mathfrak{X}$ goes into the boundary $\frac{1}{2}(y_1 + y_2) = 0$ of $\mathfrak{Y}$, the boundary $x_1 = 1$ of $\mathfrak{X}$ goes into the boundary $\frac{1}{2}(y_1 - y_2) = 1$ of $\mathfrak{Y}$, and the boundary $x_2 = 1$ of $\mathfrak{X}$ goes into the boundary $\frac{1}{2}(y_1 + y_2) = 1$ of $\mathfrak{Y}$. Now the transformation is one-to-one, the first partial derivatives of g_1^{-1} and g_2^{-1} are continuous, and the Jacobian is nonzero; so

$$
\begin{aligned}
f_{Y_1, Y_2}(y_1, y_2) &= |J| f_{X_1, X_2}(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)) \\
&= \tfrac{1}{2} I_{(0,1)}\left(\frac{y_1 - y_2}{2}\right) I_{(0,1)}\left(\frac{y_1 + y_2}{2}\right) \\
&= \begin{cases} \frac{1}{2} & \text{for } (y_1, y_2) \in \mathfrak{Y} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

EXAMPLE Let X_1 and X_2 be two independent standard normal rando variables. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$. Then

$$x_1 = g_1^{-1}(y_1, y_2) = \frac{y_1 y_2}{1 + y_2} \qquad \text{and} \qquad x_2 = g_2^{-1}(y_1, y_2) = \frac{y_1}{1 + y_2}.$$

$$J = \begin{vmatrix} \dfrac{y_2}{1 + y_2} & \dfrac{y_1}{(1 + y_2)^2} \\ \dfrac{1}{1 + y_2} & -\dfrac{y_1}{(1 + y_2)^2} \end{vmatrix} = -\frac{y_1(y_2 + 1)}{(1 + y_2)^3} = -\frac{y_1}{(1 + y_2)^2}.$$

$$
\begin{aligned}
&f_{Y_1, Y_2}(y_1, y_2) \\
&\quad = \frac{|y_1|}{(1 + y_2)^2} \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left[\frac{(y_1 y_2)^2}{(1 + y_2)^2} + \frac{y_1^2}{(1 + y_2)^2}\right]\right\} \\
&\quad = \frac{1}{2\pi} \frac{|y_1|}{(1 + y_2)^2} \exp\left[-\frac{1}{2} \frac{(1 + y_2^2) y_1^2}{(1 + y_2)^2}\right].
\end{aligned}
$$

To find the marginal distribution of, say, Y_2, we must integrate out y_1; that is

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{Y_1, Y_2}(y_1, y_2)\, dy_1$$

$$= \frac{1}{2\pi} \frac{1}{(1 + y_2)^2} \int_{-\infty}^{\infty} |y_1| \exp\left[-\frac{1}{2}\frac{(1 + y_2^2)y_1^2}{(1 + y_2)^2}\right] dy_1.$$

Let

$$u = \frac{1}{2}\frac{(1 + y_2^2)}{(1 + y_2)^2} y_1^2;$$

then

$$du = \frac{(1 + y_2^2)}{(1 + y_2)^2} y_1\, dy_1$$

and so

$$f_{Y_2}(y_2) = \frac{1}{2\pi} \cdot \frac{1}{(1 + y_2)^2} \cdot \frac{(1 + y_2)^2}{1 + y_2^2} (2) \int_0^{\infty} e^{-u}\, du = \frac{1}{\pi} \cdot \frac{1}{1 + y_2^2},$$

a Cauchy density. That is, the *ratio of two independent standard normal random variables has a Cauchy distribution.* ////

EXAMPLE Let X_i have a gamma density with parameters n_i and λ for $i = 1, 2$. Assume that X_1 and X_2 are independent. Again, we seek the joint distribution of $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$.

$$x_1 = g_1^{-1}(y_1, y_2) = \frac{y_1 y_2}{1 + y_2} \qquad \text{and} \qquad x_2 = g_2^{-1}(y_1, y_2) = \frac{y_1}{1 + y_2};$$

$$|J| = \frac{y_1}{(1 + y_2)^2}$$

$$f_{Y_1, Y_2}(y_1, y_2)$$

$$= \frac{y_1}{(1 + y_2)^2} \cdot \frac{1}{\Gamma(n_1)} \cdot \frac{1}{\Gamma(n_2)} \lambda^{n_1 + n_2} \left(\frac{y_1 y_2}{1 + y_2}\right)^{n_1 - 1} \left(\frac{y_1}{1 + y_2}\right)^{n_2 - 1} e^{-\lambda y_1} I_{\mathfrak{P}}(y_1, y_2)$$

$$= \frac{\lambda^{n_1 + n_2}}{\Gamma(n_1)\Gamma(n_2)} y_1^{n_1 + n_2 - 1} e^{-\lambda y_1} \frac{y_2^{n_1 - 1}}{(1 + y_2)^{n_1 + n_2}} I_{(0, \infty)}(y_1) I_{(0, \infty)}(y_2)$$

$$= \left[\frac{\lambda^{n_1 + n_2}}{\Gamma(n_1 + n_2)} y_1^{n_1 + n_2 - 1} e^{-\lambda y_1} I_{(0, \infty)}(y_1)\right]$$

$$\times \left[\frac{1}{\mathrm{B}(n_1, n_2)} \frac{y_2^{n_1 - 1}}{(1 + y_2)^{n_1 + n_2}} I_{(0, \infty)}(y_2)\right].$$

We see that $f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2)$; so Y_1 and Y_2 are independent. Also, we see that the distribution of $Y_1 = X_1 + X_2$ is a gamma distribution with parameters $n_1 + n_2$ and λ. If $n_1 = n_2 = 1$, then Y_2 is the ratio of two independent exponentially distributed random variables and has density

$$f_{Y_2}(y_2) = \frac{1}{(1 + y_2)^2} I_{(0,\infty)}(y_2),$$

a density which has an infinite mean. ////

Theorem Let $X_1, X_2, \ldots, X_n$ be jointly continuous random variables with density function $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$. Let $\mathfrak{X} = \{(x_1, \ldots, x_n): f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) > 0\}$. Assume that $\mathfrak{X}$ can be decomposed into sets $\mathfrak{X}_1, \ldots, \mathfrak{X}_m$ such that $y_1 = g_1(x_1, \ldots, x_n)$, $y_2 = g_2(x_1, \ldots, x_n)$, $\ldots$, $y_n = g_n(x_1, \ldots, x_n)$ is a one-to-one transformation of $\mathfrak{X}_i$ onto $\mathfrak{Y}$, $i = 1, \ldots, m$. Let $x_1 = g_{1i}^{-1}(y_1, \ldots, y_n)$, $\ldots$, $x_n = g_{ni}^{-1}(y_1, \ldots, y_n)$ denote the inverse transformation of $\mathfrak{Y}$ onto $\mathfrak{X}_i$, $i = 1, \ldots, m$. Define

$$J_i = \begin{vmatrix} \frac{\partial g_{1i}^{-1}}{\partial y_1} & \frac{\partial g_{1i}^{-1}}{\partial y_2} & \cdots & \cdots & \frac{\partial g_{1i}^{-1}}{\partial y_n} \\ \frac{\partial g_{2i}^{-1}}{\partial y_1} & \frac{\partial g_{2i}^{-1}}{\partial y_2} & \cdots & \cdots & \frac{\partial g_{2i}^{-1}}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_{ni}^{-1}}{\partial y_1} & \frac{\partial g_{ni}^{-1}}{\partial y_2} & \cdots & \cdots & \frac{\partial g_{ni}^{-1}}{\partial y_n} \end{vmatrix}$$

for $i = 1, \ldots, m$.

Assume that all the partial derivatives in J_i are continuous over $\mathfrak{Y}$ and the determinant J_i is nonzero, $i = 1, \ldots, m$. Then

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)$$
$$= \sum_{i=1}^{m} |J_i| f_{X_1,\ldots,X_n}(g_{1i}^{-1}(y_1, \ldots, y_n), \ldots, g_{ni}^{-1}(y_1, \ldots, y_n)) \tag{42}$$

for $(y_1, \ldots, y_n)$ in $\mathfrak{Y}$. ////