## Common Functions

A variety of important types of functions are frequently encountered in calculus. We identify and briefly describe them here.

**Linear Functions** A function of the form $f(x) = mx + b$, for constants $m$ and $b$, is called a **linear function**. Figure 1.14a shows an array of lines $f(x) = mx$ where $b = 0$, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function**. Constant functions result when the slope $m = 0$ (Figure 1). A linear function with positive slope whose graph passes through the origin is called a *proportionality* relationship.

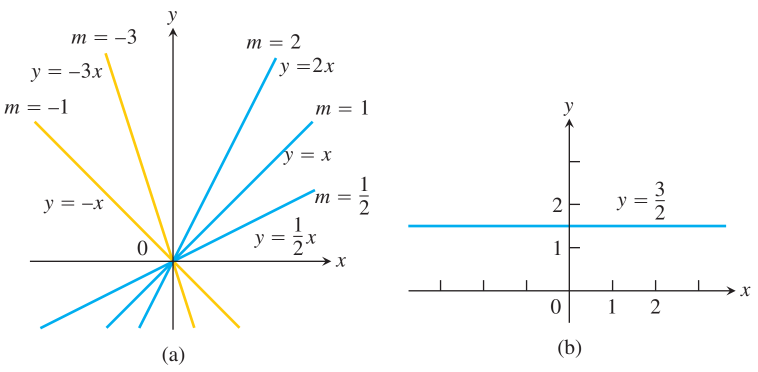



**FIGURE 1** (a) Lines through the origin with slope $m$. (b) A constant function with slope $m = 0$.

> **DEFINITION** Two variables $y$ and $x$ are **proportional** (to one another) if one is always a constant multiple of the other; that is, if $y = kx$ for some nonzero constant $k$.

If the variable $y$ is proportional to the reciprocal $1/x$, then sometimes it is said that $y$ is **inversely proportional** to $x$ (because $1/x$ is the multiplicative inverse of $x$).

**Power Functions** A function $f(x) = x^a$, where $a$ is a constant, is called a **power function**. There are several important cases to consider.

**(a)** $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 2. These functions are defined for all real values of $x$. Notice that as the power $n$ gets larger, the curves tend to flatten toward the $x$-axis on the interval $(-1, 1)$, and also rise more steeply for $|x| > 1$. Each curve passes through the point (1, 1) and through the origin. The graphs of functions with even powers are symmetric about the $y$-axis; those with odd powers are symmetric about the origin. The even-powered functions are decreasing on the interval $(-\infty, 0]$ and increasing on $[0, \infty)$; the odd-powered functions are increasing over the entire real line $(-\infty, \infty)$.

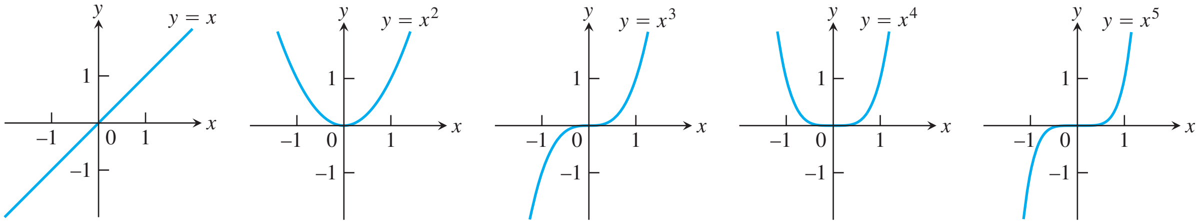



**FIGURE 2** Graphs of $f(x) = x^n, n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

**(b)** $a = -1$ or $a = -2$.

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $g(x) = x^{-2} = 1/x^2$ are shown in Figure 3. Both functions are defined for all $x \neq 0$ (you can never divide by zero). The graph of $y = 1/x$ is the hyperbola $xy = 1$, which approaches the coordinate axes far from the origin. The graph of $y = 1/x^2$ also approaches the coordinate axes. The graph of the function $f$ is symmetric about the origin; $f$ is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. The graph of the function $g$ is symmetric about the $y$-axis; $g$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

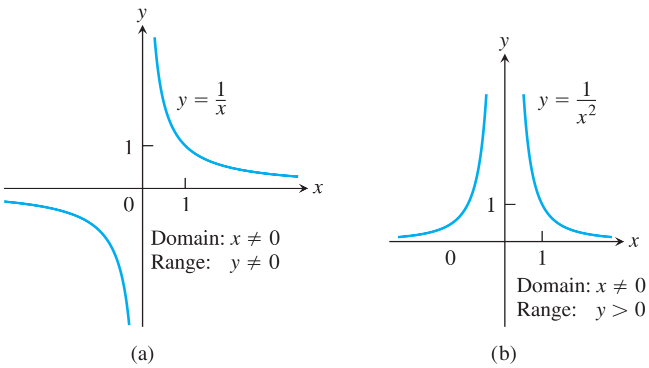



**FIGURE 3** Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

**(c)** $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real $x$. Their graphs are displayed in Figure 1.17 along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

## Composite Functions

Composition is another method for combining functions.

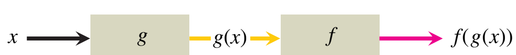
**DEFINITION Composition of Functions**

If $f$ and $g$ are functions, the **composite** function $f \circ g$ ("$f$ composed with $g$") is defined by

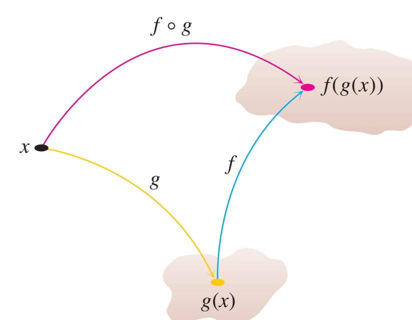
$$(f \circ g)(x) = f(g(x)).$$

$f \circ g$ consists of the numbers $x$ in the domain of $g$ for which $g(x)$ lies in the domain of $f$.

The definition says that $f \circ g$ can be formed when the range of $g$ lies in the domain of $f$. To find $(f \circ g)(x)$, *first* find $g(x)$ and *second* find $f(g(x))$.




Two functions can be composed at $x$ whenever the value of one function at $x$ lies in the domain of the other. The composite is denoted by $f \circ g$.




Arrow diagram for $f \circ g$.

**EXAMPLE** Viewing a Function as a Composite The

function $y = \sqrt{1 - x^2}$ can be thought of as first calculating $1 - x^2$ and then taking the square root of the result. The function $y$ is the composite of the function $g(x) = 1 - x^2$ and the function $f(x) = \sqrt{x}$. Notice that $1 - x^2$ cannot be negative. The domain of the composite is $[-1, 1]$. ■

To evaluate the composite function $g \circ f$ (when defined), we reverse the order, finding $f(x)$ first and then $g(f(x))$. The domain of $g \circ f$ is the set of numbers $x$ in the domain of $f$ such that $f(x)$ lies in the domain of $g$.

The functions $f \circ g$ and $g \circ f$ are usually quite different.

**EXAMPLE** Finding Formulas for Composites If

and $f(x) = \sqrt{x}$  $g(x) = x + 1$, find

**(a)** $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

**Solution**

| | **Composite** | **Domain** |
|---|---|---|
| **(a)** | $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x + 1}$ | $[-1, \infty)$ |
| **(b)** | $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$ | $[0, \infty)$ |
| **(c)** | $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$ | $[0, \infty)$ |
| **(d)** | $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$ | $(-\infty, \infty)$ |

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real $x$ but belongs to the domain of $f$ only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = \left(\sqrt{x}\right)^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$.

## Shifting a Graph of a Function

To shift the graph of a function $y = f(x)$ straight up, add a positive constant to the right-hand side of the formula $y = f(x)$.

To shift the graph of a function $y = f(x)$ straight down, add a negative constant to the right-hand side of the formula $y = f(x)$.

To shift the graph of $y = f(x)$ to the left, add a positive constant to $x$. To shift the graph of $y = f(x)$ to the right, add a negative constant to $x$.

### Shift Formulas

**Vertical Shifts**

| | |
|---|---|
| $y = f(x) + k$ | Shifts the graph of $f$ *up* $k$ units if $k > 0$ |
| | Shifts it *down* $\lvert k \rvert$ units if $k < 0$ |

**Horizontal Shifts**

| | |
|---|---|
| $y = f(x + h)$ | Shifts the graph of $f$ *left* $h$ units if $h > 0$ |
| | Shifts it *right* $\lvert h \rvert$ units if $h < 0$ |

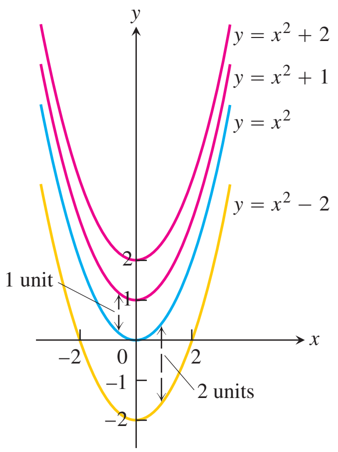



**FIGURE a.b** To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for $f$

**EXAMPLE** Shifting a Graph

**(a)** Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure a.b).

**(b)** Adding $-2$ to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure a.b).

**(c)** Adding 3 to $x$ in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left (Figure c).

**(d)** Adding $-2$ to $x$ in $y = |x|$, and then adding $-1$ to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure d).

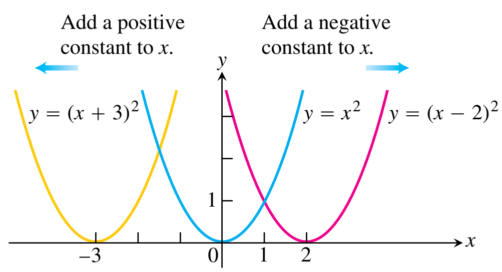



**FIGURE c** To shift the graph of $y = x^2$ to the left, we add a positive constant to $x$. To shift the graph to the right, we add a negative constant to $x$.

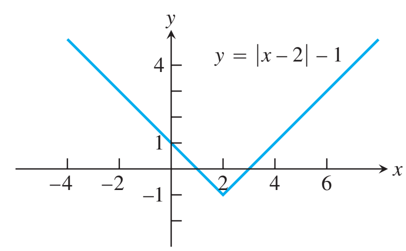



**FIGURE d** Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down.

■

## Scaling and Reflecting a Graph of a Function

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function $f$, or the independent variable $x$, by an appropriate constant $c$. Reflections across the coordinate axes are special cases where $c = -1$.

**Vertical and Horizontal Scaling and Reflecting Formulas**

For $c > 1$,

| | |
|---|---|
| $y = cf(x)$ | Stretches the graph of $f$ vertically by a factor of $c$. |
| $y = \frac{1}{c} f(x)$ | Compresses the graph of $f$ vertically by a factor of $c$. |
| $y = f(cx)$ | Compresses the graph of $f$ horizontally by a factor of $c$. |
| $y = f(x/c)$ | Stretches the graph of $f$ horizontally by a factor of $c$. |

For $c = -1$,

| | |
|---|---|
| $y = -f(x)$ | Reflects the graph of $f$ across the $x$-axis. |
| $y = f(-x)$ | Reflects the graph of $f$ across the $y$-axis. |

## EXAMPLE Scaling and Reflecting a Graph

**(a)** **Vertical:** Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get $y = 3\sqrt{x}$ stretches the graph vertically by a factor of 3, whereas multiplying by 1/3 compresses the graph by a factor of 3 (Figure a).

**(b)** **Horizontal:** The graph of $y = \sqrt{3x}$ is a horizontal compression of the graph of $y = \sqrt{x}$ by a factor of 3, and $y = \sqrt{x/3}$ is a horizontal stretching by a factor of 3 (Figure b). Note that $y = \sqrt{3x} = \sqrt{3}\sqrt{x}$ so a horizontal compression *may* correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.

**(c)** **Reflection:** The graph of $y = -\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the $x$-axis, and $y = \sqrt{-x}$ is a reflection across the $y$-axis (Figure c).

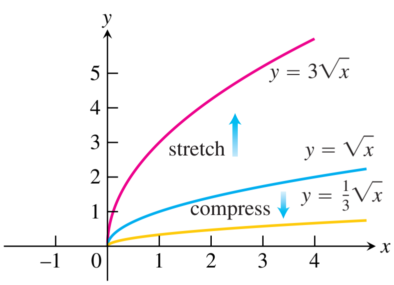



**FIGURE a** Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 .

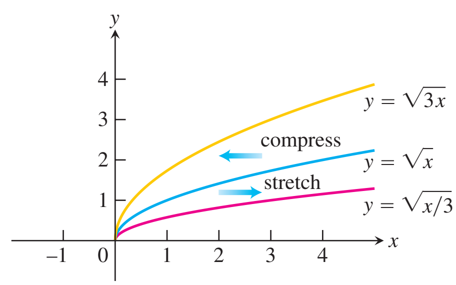



**FIGURE b** Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 .

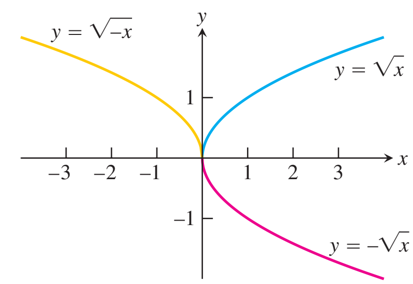



**FIGURE c** Reflections of the graph $y = \sqrt{x}$ across the coordinate axes

■

## EXAMPLE Combining Scalings and Reflections

Given the function $f(x) = x^4 - 4x^3 + 10$ (Figure a), find formulas to

**(a)** compress the graph horizontally by a factor of 2 followed by a reflection across the $y$-axis (Figure b).

**(b)** compress the graph vertically by a factor of 2 followed by a reflection across the $x$-axis (Figure c).

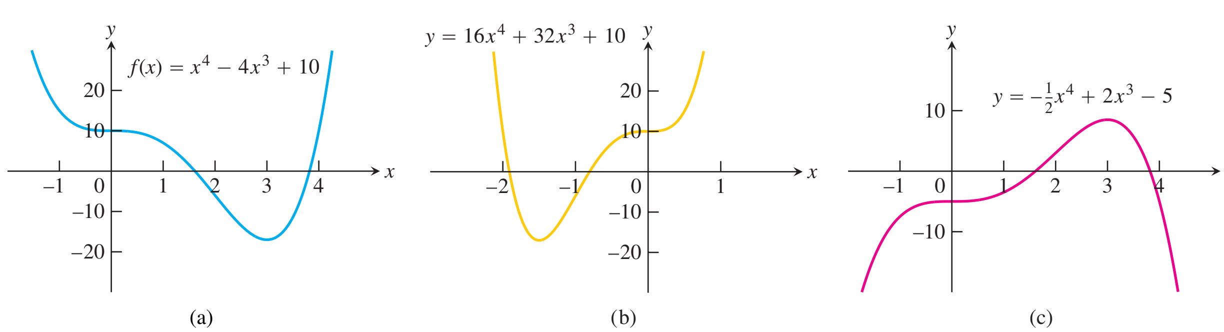



**FIGURE** (a) The original graph of $f$. (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the $y$-axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the $x$-axis

**Solution**

**(a)** The formula is obtained by substituting $-2x$ for $x$ in the right-hand side of the equation for $f$

$$\begin{aligned} y = f(-2x) &= (-2x)^4 - 4(-2x)^3 + 10 \\ &= 16x^4 + 32x^3 + 10. \end{aligned}$$

**(b)** The formula is

$$y = -\frac{1}{2} f(x) = -\frac{1}{2} x^4 + 2x^3 - 5.$$

■

You Try It

H.W

## Composites of Functions

**1.** If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

**a.** $f(g(0))$ **b.** $g(f(0))$

**c.** $f(g(x))$ **d.** $g(f(x))$

**e.** $f(f(-5))$ **f.** $g(g(2))$

**g.** $f(f(x))$ **h.** $g(g(x))$

**2.** If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.

**a.** $f(g(1/2))$ **b.** $g(f(1/2))$

**c.** $f(g(x))$ **d.** $g(f(x))$

**e.** $f(f(2))$ **f.** $g(g(2))$

**g.** $f(f(x))$ **h.** $g(g(x))$

**3.** If $u(x) = 4x - 5$, $v(x) = x^2$, and $f(x) = 1/x$, find formulas for the following.

**a.** $u(v(f(x)))$ **b.** $u(f(v(x)))$

**c.** $v(u(f(x)))$ **d.** $v(f(u(x)))$

**e.** $f(u(v(x)))$ **f.** $f(v(u(x)))$

**4.** If $f(x) = \sqrt{x}$, $g(x) = x/4$, and $h(x) = 4x - 8$, find formulas for the following.

**a.** $h(g(f(x)))$ **b.** $h(f(g(x)))$

**c.** $g(h(f(x)))$ **d.** $g(f(h(x)))$

**e.** $f(g(h(x)))$ **f.** $f(h(g(x)))$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 9 and 10 as a composite involving one or more of $f$, $g$, $h$, and $j$.

**5.** **a.** $y = \sqrt{x} - 3$ **b.** $y = 2\sqrt{x}$

**c.** $y = x^{1/4}$ **d.** $y = 4x$

**e.** $y = \sqrt{(x-3)^3}$ **f.** $y = (2x - 6)^3$

**6.** **a.** $y = 2x - 3$ **b.** $y = x^{3/2}$

**c.** $y = x^9$ **d.** $y = x - 6$

**e.** $y = 2\sqrt{x - 3}$ **f.** $y = \sqrt{x^3 - 3}$

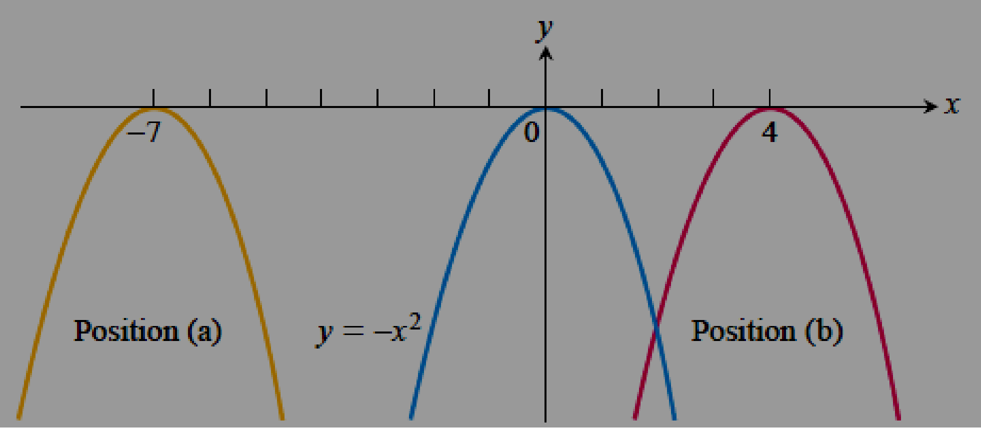
write a formula for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

7. $f(x) = \sqrt{x-1}$ , $g(x) = \frac{1}{x}$ .

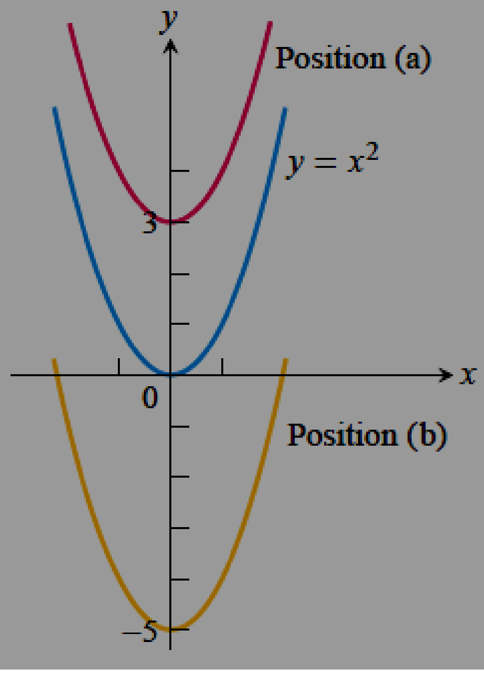
8. $f(x) = x^2$ , $g(x) = 1 - \sqrt{x}$ .

## Shifting Graphs

**9.** The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.



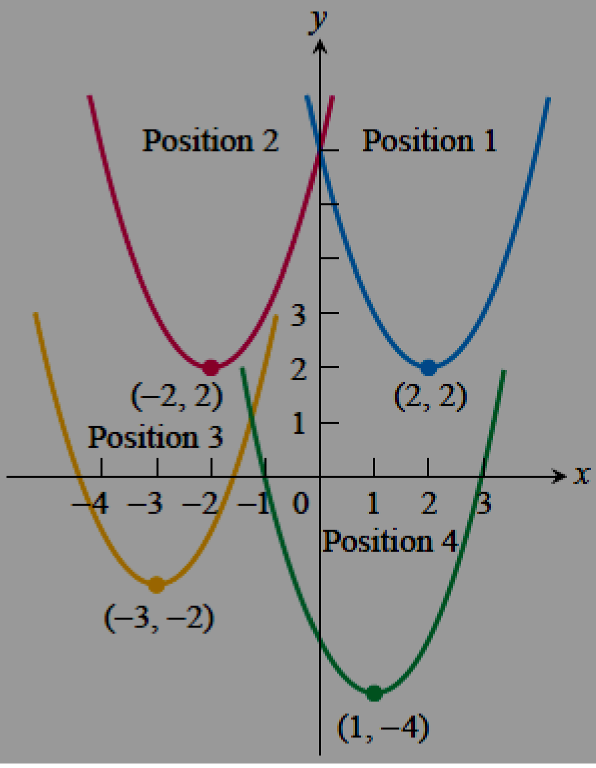
**10.** The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.



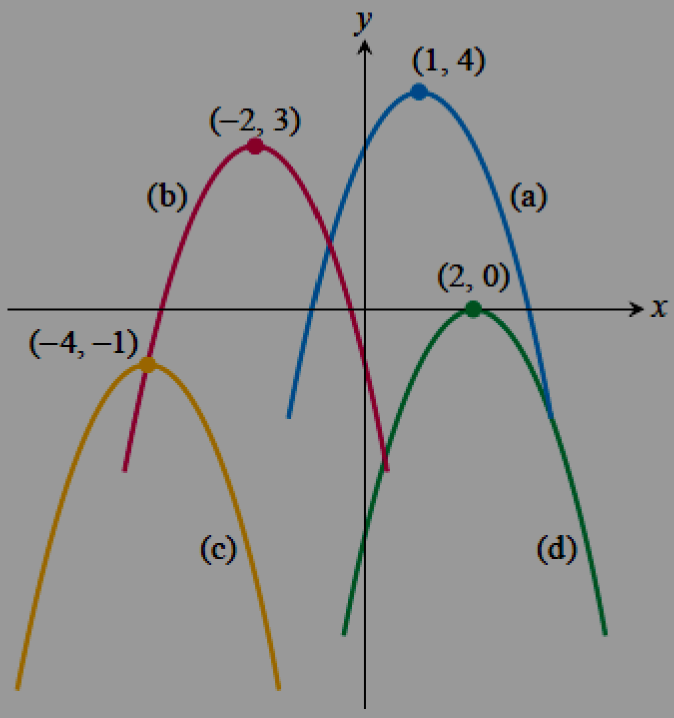

**11.** Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

**a.** $y = (x - 1)^2 - 4$ **b.** $y = (x - 2)^2 + 2$

**c.** $y = (x + 2)^2 + 2$ **d.** $y = (x + 3)^2 - 2$



**12.** The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.




Tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

**13.** $x^2 + y^2 = 49$ Down 3, left 2

**14.** $x^2 + y^2 = 25$ Up 3, left 4

**15.** $y = x^3$ Left 1, down 1

**16.** $y = x^{2/3}$ Right 1, down 1

**17.** $y = \sqrt{x}$ Left 0.81

**18.** $y = -\sqrt{x}$ Right 3

**19.** $y = 2x - 7$ Up 7

**20.** $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1

**21.** $y = 1/x$ Up 1, right 1

**22.** $y = 1/x^2$ Left 2, down 1

**Graph the functions**

**21.** $y = \sqrt{x + 4}$

**22.** $y = |x - 2|$

**23.** $y = 1 + \sqrt{x - 1}$

**24.** $y = \dfrac{1}{x - 2}$

**25.** $y = \dfrac{1}{x} + 2$

**26.** $y = \dfrac{1}{(x - 1)^2}$

**27.** $y = \dfrac{1}{x^2} + 1$