Lecture part **2** */ Assist. Prof. Dr. Kareema Abed AL_Kadim*

Convergence in Distribution

Definition

Suppose that X_n, $n \in \mathbb{N}_+$ and X are real-valued random variables with distribution functions F_n, $n \in \mathbb{N}_+$ and F, respectively. We say that the distribution of X_n **converges** to the distribution of X as $n \to \infty$ if

$$F_n(x) \to F(x) \text{ as } n \to \infty$$

for all x at which F is continuous. The first fact to notice is that convergence in distribution, as the name suggests, only involves the *distributions* of the random variables. Thus, the random variables need not even be defined on the same probability space (that is, they need not be defined for the same random experiment). This is in sharp contrast to the other modes of convergence we have studied:

Probability Density Functions

it is quite possible to have a sequence of discrete distributions converge to a continuous distribution (or the other way around). Recall that probability density functions have very different meanings in the discrete and continuous cases: density with respect to counting measure in the first case, and density with respect to Lebesgue measure in the second case. This is another indication that distribution functions, rather than density functions, are the correct objects of study. However, if probability density functions of a fixed type converge then the distributions converge. The following results are a consequence of Scheffe's theorem, which is given in advanced topics below.

Suppose that f_n, $n \in \mathbb{N}_+$ and f are probability density functions for discrete distributions on a countable set S, and that $f_n(x) \to f(x)$ as $n \to \infty$ for each $x \in S$. Then the distribution defined by f_n converges to the distribution defined by f as $n \to \infty$. Similarly, suppose that f_n, $n \in \mathbb{N}_+$ and f are probability density functions for continuous distributions on $\mathbb{R}$, and that $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in \mathbb{R}$ (except perhaps on a set with Lebesgue measure 0). Then the distribution defined by f_n converges to the distribution defined by f as $n \to \infty$.

Example (Law of rare events) Suppose there are totally n flights worldwide each year, and each flight has chance p_n to have an accident, independent of rest flights. There is on average λ accidents a year worldwide. The distribution of the number of accidents is $B(n, p_n)$ with np_n close to λ. Then this distribution approximates Poisson distribution with mean λ, namely,

$$Bin(n, p_n) \to \mathcal{P}(\lambda) \quad \text{if} \quad n \to \infty \text{ and } np_n \to \lambda > 0.$$

Proof. For any fixed $k \geq 0$, and $n \geq k$

$$\begin{aligned} P(Bin(n,p_n)=k) &= \binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{n!}{k!(n-k)!} \frac{(np_n)^k}{n^k} \frac{(1-p_n)^n}{(1-p_n)^k} \\ &= \frac{1}{k!} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{(np_n)^k e^{n\log(1-p_n)}}{(1-p_n)^k} \\ &\to \frac{\lambda^k e^{-\lambda}}{k!}, \qquad \text{as } n \to \infty. \end{aligned}$$

Remark 5.1 *The r.v. X is said to have a degenerate p.d.f. at point $X = c$ (or X is degenerate r.v. at point $X = c$) if the p.d.f. of X is*

$$f(x) = \begin{cases} 1, & \text{if } x = c; \\ 0, & \text{if } x \neq c. \end{cases} \equiv X \sim Deg(c)$$

The d.f. of X is

$$F(x) = \begin{cases} 0, & \text{if } y < c; \\ 1, & \text{if } y \geq c. \end{cases}$$

Definition 5.1 *Let $X_n \sim F_n(x)$ which depends upon n. If*

$$\lim_{n \longrightarrow \infty} F_n(x) = F(x)$$

for every point x at which $F(x)$ is continuous then $F(x)$ is the limiting distribution (l.d.) of $F_n(x)$

Remark 5.2 *Let $X_n \sim F_n(x)$ and $X \sim F(x)$. If $F_n(x) \longrightarrow F(x)$ for every point x at which $F(x)$ is continuous then, the sequence of r.v.'s X_n converges in distribution to the r.v. X and denoted by $X_n \xrightarrow{D} X$.*

Example 5.2 *Let $X_1, X_2, \cdots, X_n$ be a r.s. from $U(0,\theta)$. Let $Y_n = \max(X_1, X_2, \cdots, X_n)$. Find the l.d. of Y_n.*

Solution :

$$f(x) = \begin{cases} \frac{1}{\theta}, & 0 < x < \theta;\ 0 < \theta; \\ 0, & \text{elsewhere.} \end{cases}$$

$$\Longrightarrow F(x) = \frac{x}{\theta}$$

$$\Longrightarrow g_n(y) = \begin{cases} \frac{ny^{n-1}}{\theta^n}, & 0 < y < \theta; \\ 0, & \text{elsewhere.} \end{cases}$$

So

$$G_n(y) = \mathbb{P}(Y_n \leq y) = \int_0^y \frac{nz^{n-1}}{\theta^n} dz = \begin{cases} 0, & y < 0; \\ \left(\frac{y}{\theta}\right)^n, & 0 \leq y < \theta; \\ 1, & y \geq \theta. \end{cases}$$

Then

$$\lim_{n\to\infty} G_n(y) = \begin{cases} 0, & y < \theta; \\ 1, & y \geq \theta. \end{cases} = Deg(\theta)$$

Hence

$$G_n(y) \longrightarrow Deg(\theta) \text{ as } n \longrightarrow \infty.$$

Let

$$Y \sim G(y) = \begin{cases} 0, & y < \theta; \\ 1, & y \geq \theta. \end{cases} \Longrightarrow g(y) = \begin{cases} 1, & \text{if } y = \theta; \\ 0, & \text{if } y \neq \theta. \end{cases}$$

i.e. $Y \sim Deg(\theta)$. Then $Y_n \xrightarrow{D} Y$.

Example 5.3 *Let* $X_1, X_2, \cdots, X_n \sim^{r.s.} N(0,1)$. *Show that*

$$\bar{X}_n \xrightarrow{D} \bar{X}$$

where $\bar{X} \sim Deg(0)$.

Solution : Since

$$\because X_i \sim N(0,1) \Longrightarrow \bar{X}_n \sim N(0, 1/n) \Longrightarrow F_n(\bar{x}) = \int_{-\infty}^{\bar{x}} \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{n}{2}w^2} dw.$$

Let $Y = \sqrt{n}W$ then

$$F_n(\bar{x}) = \int^{\sqrt{n}\bar{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy.$$

$$\lim_{n\to\infty} F_n(\bar{x}) = \lim_{n\to\infty} \int_{-\infty}^{\sqrt{n}\bar{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy$$

$$= \begin{cases} \int_{-\infty}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy, & \bar{x} < 0; \\ \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy, & \bar{x} = 0; \\ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy, & \bar{x} > 0. \end{cases} = \begin{cases} 0, & \text{if } \bar{x} < 0; \\ 1/2, & \text{if } \bar{x} = 0, \\ 1, & \text{if } \bar{x} > 0. \end{cases}$$

Let

$$\bar{X} \sim F(\bar{x}) = \begin{cases} 0, & \text{if } \bar{x} < 0; \\ 1, & \text{if } \bar{x} \geq 0. \end{cases} \Rightarrow f(\bar{x}) = \begin{cases} 1, & \text{if } \bar{x} = 0; \\ 0, & \text{if } \bar{x} \neq 0. \end{cases}$$

i.e. $\bar{X} \sim Deg(0)$. Note that $F(\bar{x})$ is not continuous function at $\bar{x} = 0$. Then

$$\lim_{n\to\infty} F_n(\bar{x}) = F(\bar{x}), \forall \bar{x} \neq 0 \Longrightarrow \bar{X}_n \xrightarrow{D} \bar{X}.$$

Example 5.5 *Let* $X_1, X_2, \cdots, X_n \sim^{r.s.} U(0,\theta)$. *Find the l.d. of* $Z_n = n(\theta - Y_n)$, *where* $Y_n = \max\{X_1, \ldots, X_n\}$.

Solution : Since

$$f(x)=\begin{cases}\frac{1}{\theta}, & 0<x<\theta;\\ 0, & \text{elsewhere.}\end{cases}$$

$$Y_n \sim g_n(y)=\begin{cases}\frac{ny^{n-1}}{\theta^n}, & 0<y<\theta;\\ 0, & \text{elsewhere.}\end{cases}$$

Let $Z_n = n(\theta - Y_n) \Longrightarrow dZ_n = -ndY_n$. So that

$$Z_n \sim h_n(z)=\begin{cases}\frac{(\theta-z/n)^{n-1}}{\theta^n}, & 0<z<n\theta;\\ 0, & \text{elsewhere.}\end{cases}$$

$$H_n(z)=\begin{cases}0, & z<0;\\ \int_0^z \frac{(\theta-t/n)^{n-1}}{\theta^n}dt, & 0\le z<n\theta;\\ 1, & z\ge n\theta.\end{cases}$$

$$=\begin{cases}0, & z<0;\\ 1-\left(1-\frac{z}{n\theta}\right)^n, & 0\le z<n\theta;\\ 1, & z\ge n\theta\end{cases}$$

Since

$$\boxed{\lim_{n\longrightarrow\infty}\left(1+\frac{t}{n}\right)^n = e^t = \lim_{n\longrightarrow 0}(1+nt)^{\frac{1}{n}}}$$

$$H_n(z) = \begin{cases} 0, & z < 0; \\ \int_0^z \frac{(\theta - t/n)^{n-1}}{\theta^n} dt, & 0 \le z < n\theta; \\ 1, & z \ge n\theta. \end{cases}$$

$$= \begin{cases} 0, & z < 0; \\ 1 - \left(1 - \frac{z}{n\theta}\right)^n, & 0 \le z < n\theta; \\ 1, & z \ge n\theta \end{cases}$$

$$\boxed{\lim_{n \longrightarrow \infty} \left(1 + \frac{t}{n}\right)^n = e^t = \lim_{n \longrightarrow 0} (1 + nt)^{\frac{1}{n}}}$$

Hence

$$\lim_{n \to \infty} H_n(z) = \begin{cases} 0, & z < 0; \\ 1 - e^{-z/\theta}, & z \ge 0; \end{cases}$$

$$Z \sim H(z) = \begin{cases} 0, & z < 0; \\ 1 - e^{-z/\theta}, & z \geq 0; \end{cases}$$

$$\Longrightarrow h(z) = \begin{cases} \frac{1}{\theta}e^{-z/\theta}, & z \geq 0; \\ 0, & z < 0; \end{cases}$$

i.e. $Z \sim E(\theta)$.

$H(z)$ is the l.d. of Z_n or $Z_n \xrightarrow{D} Z$, where $Z \sim E(\theta)$.

Homework :

Q(5.1): Let $X_1, \ldots, X_n \sim^{r.s.} N(\mu, \sigma^2)$. Find the l.d. of $\bar{X}$.

Q(5.2): Let $X_1, \ldots, X_n \sim^{r.s.} f(x) = e^{-(x-\theta)}, \theta < x < \infty$, zero elsewhere. Find the l.d. of $Z_n = n(Y_1 - \theta)$, where $Y_1 = \min(X_1, \ldots, X_n)$.

Q(5.3): Let $X_1, \ldots, X_n \sim^{r.s.}$ a continuous p.d.f. $f(x)$ with d.f $F(x)$. Find the l.d. of $Z_n = n(1 - F(Y_n))$, where $Y_n = \max(X_1, \ldots, X_n)$.

Q(5.4): Let $Y_n \sim f_n(y) = 1, y = 1$, zero elsewhere. Show that Y_n does not have a l.d.

Q(5.5): Let $X_1, \ldots, X_n \sim^{r.s.}$ a continuous p.d.f. $f(x)$ with d.f $F(x)$. Find the l.d. of $Z_n = nF(Y_2)$, where $Y_2 = 2nd\ \min(X_1, \ldots, X_n)$.

Q(5.6): Let $X_1, \ldots, X_n \sim^{r.s.} N(\mu, \sigma^2)$. Show that $Z_n = \sum_{i=1}^{n} X_i$ does not have a l.d.

5.2 Convergence in Probability

Definition 5.2 *A sequence of r.v.'s* $X_1, X_2, \cdots, X_n$ *converges in probability to a r.v.* X *if* $\forall \epsilon > 0$,

$$\lim_{n \longrightarrow \infty} \mathbb{P}(|X_n - X| < \epsilon) = 1$$

or equivalently

$$\lim_{n \longrightarrow \infty} \mathbb{P}(|X_n - X| > \epsilon) = 0$$

and it will be denoted by

$$X_n \xrightarrow{P} X$$

Theorem 5.1 *Chebyshev's Inequality*

Let Y *be a r.v. that has mean* μ *and finite variance* σ^2. *Then for every* $k > 0$,

$$\mathbb{P}(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

or equivalently

$$\mathbb{P}(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Theorem 5.2 *Weak Law of Large Numbers*

Let $X_1, X_2, \cdots, X_n$ be a r.s. from a distribution that has mean μ and variance σ^2. Show that $\bar{X}_n \xrightarrow{P} \mu$.

Proof : It is known that $\mathbb{E}(\bar{X}) = \mu$ and $\mathbb{V}ar(\bar{X}) = \sigma^2/n$. Consider $\forall \epsilon > 0$, and let $k = \dfrac{\epsilon\sqrt{n}}{\sigma}$.

$$\mathbb{P}(|\bar{X}_n - \mu| > \epsilon) = \mathbb{P}\left(|\bar{X}_n - \mu| > \frac{\epsilon\sqrt{n}}{\sigma}\frac{\sigma}{\sqrt{n}}\right) = \mathbb{P}\left(|\bar{X}_n - \mu| > \frac{k\sigma}{\sqrt{n}}\right).$$

From Chebyshev's inequality

$$\mathbb{P}\left(|\bar{X}_n - \mu| > \frac{k\sigma}{\sqrt{n}}\right) \leq \frac{1}{k^2} = \frac{\sigma^2}{n\epsilon^2}.$$

$$\lim_{n\longrightarrow\infty} \mathbb{P}(|\bar{X}_n - \mu| > \epsilon) \leq \lim_{n\longrightarrow\infty} \frac{\sigma^2}{n\epsilon^2} = \frac{\sigma^2}{\epsilon^2} \lim_{n\longrightarrow\infty} \frac{1}{n} = 0.$$

Then $\bar{X}_n \xrightarrow{P} \mu$.

Example 5.6 *(Q:5.7) Let $Y_n \sim B(n,p)$. Prove that*

(i) $\frac{1}{n}Y_n \xrightarrow{P} p$.

(ii) $1 - \frac{1}{n}Y_n \xrightarrow{P} 1 - p$.

Solution : (i) Note that

$$\mathbb{E}(Y_n) = np \Longrightarrow \mathbb{E}\left(\frac{Y_n}{n}\right) = p$$

and

$$\mathbb{V}ar\left(\frac{Y_n}{n}\right) = \frac{1}{n^2}\mathbb{V}ar(Y_n) = \frac{1}{n^2}np(1-p) = \frac{p(1-p)}{n}.$$

Suppose for every $\epsilon > 0$ and let $k = \epsilon\sqrt{\frac{n}{p(1-p)}}$.

$$\begin{aligned}\mathbb{P}\left(\left|\frac{1}{n}Y_n - p\right| > \epsilon\right) &= \mathbb{P}\left(\left|\frac{1}{n}Y_n - p\right| > \epsilon\frac{\sqrt{np(1-p)}}{\sqrt{np(1-p)}}\right)\\ &= \mathbb{P}\left(\left|\frac{1}{n}Y_n - p\right| > k\sigma\right)\\ &\le \frac{1}{k^2}\\ &= \frac{p(1-p)}{n\epsilon}.\end{aligned}$$

$$\lim_{n\longrightarrow\infty} \mathbb{P}\left(\left|\frac{1}{n}Y_n - p\right| > \epsilon\right) \leq \lim_{n\longrightarrow\infty} \frac{p(1-p)}{n\epsilon} = 0.$$

(ii)

$$\lim_{n\longrightarrow\infty} \mathbb{P}\left(\left|1 - \frac{1}{n}Y_n - 1 + p\right| > \epsilon\right) = \lim_{n\longrightarrow\infty} \mathbb{P}\left(\left|p - \frac{1}{n}Y_n\right| > \epsilon\right)$$
$$= \lim_{n\longrightarrow\infty} \mathbb{P}\left(\left|\frac{1}{n}Y_n - p\right| > \epsilon\right) = 0.$$

Example 5.7 *(Q:5.8) Let* $X_1, X_2, \cdots, \cdots, X_n \sim^{r.s.} N(\mu, \sigma^2)$. *Prove that*

$$\boxed{\frac{nS_n^2}{n-1} \xrightarrow{P} \sigma^2}$$

<u>**Solution :**</u> Since

$$\frac{nS_n^2}{\sigma^2} \sim \chi^2_{n-1}$$

then

$$\mathbb{E}\left(\frac{nS_n^2}{\sigma^2}\right) = n-1 \quad \text{and} \quad \mathbb{V}ar\left(\frac{nS_n^2}{\sigma^2}\right) = 2(n-1).$$

So that

$$\mathbb{E}\left(\frac{nS_n^2}{n-1}\right) = \frac{\sigma^2}{n-1}\mathbb{E}\left(\frac{nS_n^2}{\sigma^2}\right) = \frac{\sigma^2}{n-1}(n-1) = \sigma^2$$

and

$$\mathbb{V}ar\left(\frac{nS_n^2}{n-1}\right) = \frac{\sigma^4}{(n-1)^2}\mathbb{V}ar\left(\frac{nS_n^2}{\sigma^2}\right) = \frac{\sigma^4}{(n-1)^2}2(n-1) = \frac{2\sigma^4}{n-1}.$$

Now for $\epsilon > 0$ and $k = \epsilon\sqrt{\frac{n-1}{2\sigma^4}}$ we have

$$\lim_{n\to\infty}\mathbb{P}\left(\left|\frac{nS_n^2}{n-1} - \sigma^2\right| > \epsilon\right) = \lim_{n\to\infty}\mathbb{P}\left(\left|\frac{nS_n^2}{n-1} - \sigma^2\right| > \epsilon\sqrt{\frac{n-1}{2\sigma^4}}\sqrt{\frac{2\sigma^4}{n-1}}\right)$$

$$= \lim_{n\to\infty}\mathbb{P}\left(\left|\frac{nS_n^2}{n-1} - \sigma^2\right| > k\sqrt{\frac{2\sigma^4}{n-1}}\right)$$

$$\le \lim_{n\to\infty} \frac{1}{k^2}$$

$$= \frac{2\sigma^4}{\epsilon^2} \lim_{n\to\infty} \frac{1}{n-1}$$

$$= 0.$$

Homework :

Q:5.7. Let $Y_n \sim B(n,p)$. Prove that (a) $\frac{1}{n}Y_n \xrightarrow{P} p$. (b) $1-\frac{1}{n}Y_n \xrightarrow{P} 1-p$.

Q:5.8. Let $X_1, X_2, \cdots, X_n \sim^{r.s.} N(\mu,\sigma^2)$. Prove that $\dfrac{nS_n^2}{n-1} \xrightarrow{P} \sigma^2$.

Q:5.9. Let W_n be a r.v. with mean μ and variance b/n^t where $t > 0$, μ, and b are constants (not functions of n). Prove that $W_n \xrightarrow{P} \mu$.

Q:5.10. Let $X_1, X_2, \cdots, X_n \sim^{r.s.} U(0,\theta)$. Prove that $\sqrt{Y_n} \xrightarrow{P} \sqrt{\theta}$, where $Y_n = \max(X_1, \ldots, X_n)$.

5.3 The Central Limit Theorem (CLT)

Let $X_1, X_2, \cdots, X_n \sim^{r.s.} N(\mu,\sigma^2)$. Then

$$\boxed{Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \sim N(0,1)},$$

where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Theorem 5.3 *Let $X_1, X_2, \cdots, X_n$ be a r.s. from a distribution that has mean μ and positive variance σ^2. Then*

$$Y_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \simeq^{CLT} N(0,1) \text{ as } n \longrightarrow \infty$$

Example 5.8 *Let $X_1, X_2, \cdots, X_{75}$ be a random sample from $U(0,1)$. Using the CLT, find approximately $\mathbb{P}(0.45 < \bar{X}_n < 0.55)$.*

<u>**Solution :**</u> Since

$$f(x) = \begin{cases} 1, & 0 < x < 1; \\ 0, & \text{elsewhere.} \end{cases}$$

then $\mu = 1$ and $\sigma^2 = \frac{1}{12}$.

$$\Longrightarrow Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\sqrt{75}(\bar{X}_n - 1)}{1/\sqrt{12}} = 30(\bar{X}_n - 1) \simeq^{CLT} N(0,1) \text{ as } n \longrightarrow \infty.$$

So

$$\begin{aligned} \mathbb{P}(0.45 < \bar{X}_n < 0.55) &= \mathbb{P}(30(0.45 - 1) < Z_n < 30(0.55 - 1)) \\ &= \mathbb{P}(-1.5 < Z_n < 1.5) \\ &\simeq^{CLT} 2\Phi(1.5) - 1 = 0.866. \end{aligned}$$

Example 5.9 *Let* $X_1, X_2, \cdots, X_{100} \sim^{r.s.} B(1, 0.5)$. *Using CLT, find approximately* $\mathbb{P}(Y_n = 48, 49, 50, 51, 52)$, *where* $Y_n = \sum_{i=1}^{100} X_i$.

Solution : Since

$$X_i \sim B(1, 0.5) \Longrightarrow \mu = p = 0.5$$

$$\sigma^2 = p(1-p) = (0.5)(0.5) = 0.25.$$

Therefore

$$Z_n = \frac{Y_n - n\mu}{\sqrt{n}\sigma} = \frac{Y_n - 100(0.5)}{\sqrt{100}\,0.5} = \frac{Y_n - 50}{5} \simeq^{CLT} N(0,1) \text{ as } n \longrightarrow \infty.$$

So that

$$\mathbb{P}(Y_n = 48, 49, 50, 51, 52) = \mathbb{P}(47.5 < Y_n < 52.5)$$

$$= \mathbb{P}\left(\frac{47.5 - 50}{5} < \frac{Y_n - 50}{5} < \frac{52.5 - 50}{5}\right)$$

$$= \mathbb{P}\left(-0.5 < Z_n < 0.5\right)$$

$$\simeq^{CLT} 2\Phi(0.5) - 1 = 0.382.$$

Homework :

Q:5.21. Let $X_1, \ldots, X_{100} \sim^{r.s.} \chi^2_{50}$. Compute approximately $\mathbb{P}(49 < \bar{X}_n < 51)$.

Q:5.22. Let $X_1, \ldots, X_{128} \sim^{r.s.} G(2, 4)$. Compute approximately $\mathbb{P}(7 < \bar{X}_n < 9)$.

Q:5.23. Let $X_1, \ldots, X_{72} \sim^{r.s.} B(1, \frac{1}{3})$. Let $Y_n = \sum_{i=1}^{72} X_i$. Compute approximately $\mathbb{P}(22 < Y_n < 28)$.

Q:5.24. Let $X_1, \ldots, X_{15} \sim^{r.s.} f(x) = 3x^2, 0 < x < 1$, zero elsewhere. Compute approximately $\mathbb{P}(\frac{3}{5} < \bar{X}_n < \frac{4}{5})$.

Q:5.25. Let $X_1, \ldots, X_{12} \sim^{r.s.} f(x) = \frac{1}{6}, x = 1, 2, 3, 4, 6$, zero elsewhere. Compute approximately $\mathbb{P}(36 \leq Y_n \leq 48)$, where $Y_n = \sum_{i=1}^{12} X_i$.

Q:5.27. Let $X_1, \ldots, X_{100} \sim^{r.s.} B(1, \frac{1}{2})$. Let $Y_n = \sum_{i=1}^{100} X_i$. Compute approximately $\mathbb{P}(Y_n = 50)$.

Q:5.30.(Hogg Ed. 5) Let $X_1 \ldots, X_n \sim N(\mu, \sigma^2)$. Use the Delta method to find the approximated distribution of $u(\bar{X}_n) = \bar{X}_n^3$.