The differential equations variant coefficients

A second order nonhomogeneous equation with variant coefficients is written as
$P(x)y'' + Q(x)y' + R(x)y = F(x)$

I- Euler's differential equation

We want to look for solutions to the differential equation in the form

$$ax^2y'' + bxy' + cy = f(x)$$

These type of differential equations are called *Euler equations.*

This differential equation can be transformed into a second order homogeneous equation with constant coefficients by use of the substitution: $x = e^t$.

Differentiating with respect to t gives:

$$\frac{dx}{dt} = e^t \quad \text{so} \quad \frac{dx}{dt} = x$$

In the Euler equation we see the term xy' which we need to substitute for, so we will multiply both sides of $\frac{dx}{dt} = x$ by y':

$$\frac{dy}{dx} \times \frac{dx}{dt} = x\frac{dy}{dx}$$

This simplifies to $\frac{dy}{dt} = xy'$.

Differentiating this with respect to x gives: $\frac{d}{dx}\frac{dy}{dt} = xy'' + y'$.

In the Euler equation we see the term xy' which we need to substitute for, so we will multiply the left side of the above equation by $\frac{dx}{dt}$ and the right hand side by x (Remember they are equal) gives:

$$x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

So under this substitution the Euler equation becomes:

$$\boxed{a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = f(e^t)}$$

This is a second order linear equation with constant coefficients.

Example 1: Solve $x^2y'' + xy' + y = 2$

Solution : The given differential equation is Euler equation

Put $x = e^t$

Then we get $\dfrac{d^2y}{dt^2} + (1-1)\dfrac{dy}{dt} + y = 2$

Or $\dfrac{d^2y}{dt^2} + y = 2$

$$m^2 + 1 = 0 \;\Rightarrow\; m = \pm i$$

$$y_h = c_1 \sin t + c_2 \cos t$$

Let $y_p = A$ then $\dfrac{dy}{dt} = \dfrac{d^2y}{dt^2} = 0 \;\Rightarrow\; A = 2 \;\Rightarrow\; y_p = 2$

$y = y_h + y_p = c_1 \sin t + c_2 \cos t + 2$

We have $x = e^t \;\Rightarrow\; t = \ln x$

Then $y = c_1 \sin(\ln x) + c_2 \cos(\ln x) + 2$

Example 2: Solve $x^2y'' - 2xy' + 2y = 4x^3$

Solution : The given differential equation is Euler equation

Put $x = e^t$

Then we get $\dfrac{d^2y}{dt^2} + (-2-1)\dfrac{dy}{dt} + 2y = 4e^{3t} \;\Rightarrow\; \dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 2y = 4e^{3t}$

$$m^2 - 3m + 2 = 0 \;\Rightarrow (m-1)(m-2) = 0 \;\Rightarrow\; m_1 = 1 \text{ and } m_2 = 2$$

$$y_h = c_1 e^t + c_2 e^{2t}$$

Let $y_p = Ae^{3t}$ then $\dfrac{dy}{dt} = 3Ae^{3t}$ and $\dfrac{d^2y}{dt^2} = 9Ae^{3t}$

$$9Ae^{3t} - 9Ae^{3t} + 2Ae^{3t} = 4e^{3t} \;\Rightarrow\; A = 2 \;\Rightarrow\; y_p = 2e^{3t}$$

So $y = y_h + y_p = c_1 e^{2t} + c_2 e^t + 2e^{3t}$

We have $x = e^t$

Then $y = c_1 x^2 + c_2 x + 2x^3$

Example 3: Solve $x^2y'' + 6xy' + 6y = \ln x$

Solution : The given differential equation is Euler equation

Put $x = e^t$

Then we get $\dfrac{d^2y}{dt^2} + (6-1)\dfrac{dy}{dt} + 6y = t \Rightarrow \dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 6y = t$

$m^2 + 5m + 6 = 0 \Rightarrow (m+3)(m+2) = 0 \Rightarrow m_1 = -3 \quad , \quad m_2 = -2$

$$y_h = c_1e^{-3t} + c_2e^{-2t}$$

Let $y_p = At + B$ then $\dfrac{dy}{dt} = A$ and $\dfrac{d^2y}{dt^2} = 0$

$$5A + 6At + 6B = t$$

$$6A = 1 \Rightarrow A = (1/6)$$

$$5A + 6B = 0 \Rightarrow B = -(5/36)$$

$$y_p = (1/6)t - (5/36)$$

$y = y_h + y_p = c_1e^{-3t} + c_2e^{-2t} + (1/6)t - (5/36)$

$y = c_1x^{-3} + c_2x^{-2} + (1/6)\ln x - (5/36)$

Exercises

Solve the differential equations

(1) $x^2y'' - 5xy' + 8y = 2x^2$

(2) $x^2y'' - xy' - 3y = x^5$

(3) $x^2y'' + 5xy' + 4y = \dfrac{1}{x^2}$

(4) $x^2y'' + 3xy' + 5y = \ln x$