Lecture 7

Linear programming :
Artifical variable technique :
Two - Phase method

7.1 <u>Steps for Two-Phase Method</u>

The process of eliminating artificial variables is performed in **phase-I** of the solution and **phase-II** is used to get an optimal solution. Since the solution of LPP is computed in two phases, it is called as **Two-Phase Simplex Method**.

Phase I – In this phase, the simplex method is applied to a specially constructed **auxiliary linear programming problem** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1 – Assign a cost -1 to each artificial variable and a cost 0 to all other variables in the objective function.

Step 2 – Construct the Auxiliary LPP in which the new objective function Z* is to be maximized subject to the given set of constraints.

Step 3 – Solve the auxiliary problem by simplex method until either of the following three possibilities do arise

i. Max $Z^* < 0$ and atleast one artificial vector appear in the optimum basis at a positive level ($\Delta_j \geq 0$). In this case, given problem does not possess any feasible solution.
ii. Max $Z^* = 0$ and at least one artificial vector appears in the optimum basis at a zero level. In this case proceed to phase-II.
iii. Max $Z^* = 0$ and no one artificial vector appears in the optimum basis. In this case also proceed to phase-II.

Phase II – Now assign the actual cost to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints.

Simplex method is applied to the modified simplex table obtained at the end of phase-I, until an optimum basic feasible solution has been attained. The artificial variables which are non-basic at the end of phase-I are removed.

7.2 <u>Worked Examples</u>

Example 1

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 2$

$x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 2$

$x_2 + s_3 = 4$

$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 - 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1$

Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 2$

$x_2 + s_3 = 4$

$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$

Phase I

		$C_j \rightarrow$	0	0	0	0	0	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min ratio X_B / X_k
a_1	-1	2	$\boxed{2}$	1	-1	0	0	1	$1\rightarrow$
s_2	0	2	1	3	0	1	0	0	2
s_3	0	4	0	1	0	0	1	0	-
	$Z^* = -2$		$\uparrow$ -2	-1	1	0	0	0	$\leftarrow\Delta_j$
x_1	0	1	1	1/2	-1/2	0	0	X	
s_2	0	1	0	5/2	1/2	1	0	X	
s_3	0	4	0	1	0	0	1	X	
	$Z^* = 0$		0	0	0	0	0	X	$\leftarrow\Delta_j$

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

		$C_j \rightarrow$	3	-1	0	0	0

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B/X_k
x_1	3	1	1	1/2	-1/2	0	0	-
s_2	0	1	0	5/2	$\boxed{1/2}$	1	0	2→
s_3	0	4	0	1	0	0	1	-
	Z = 3		0	5/2	↑ -3/2	0	0	←Δ_j
x_1	3	2	1	3	0	1	0	
s_1	0	2	0	5	1	2	0	
s_3	0	4	0	1	0	0	1	
	Z = 6		0	10	0	3	0	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 6, $x_1 = 2$, $x_2 = 0$

Example 2

Max $Z = 5x_1 + 8x_2$

Subject to

$3x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$x_1 + x_2 \leq 5$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

Standard LPP

Max $Z = 5x_1 + 8x_2$

Subject to

$3x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1 + x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1 - 1a_2$

Subject to

$3x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1 + x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0$

Phase I

		$C_j \rightarrow$	0	0	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	A_2	Min ratio X_B / X_k
a_1	-1	3	3	2	-1	0	0	1	0	3/2
a_2	-1	4	1	$\boxed{4}$	0	-1	0	0	1	1→
s_3	0	5	1	1	0	0	1	0	0	5
	$Z^* = -7$		-4	↑ -6	1	1	0	0	0	$\leftarrow \Delta_j$
a_1	-1	1	$\boxed{5/2}$	0	-1	1/2	0	1	X	2/5→
x_2	0	1	1/4	1	0	-1/4	0	0	X	4
s_3	0	4	3/4	0	0	1/4	1	0	X	16/3
	$Z^* = -1$		↑ -5/2	0	1	-1/2	0	0	X	$\leftarrow \Delta_j$
x_1	0	2/5	1	0	-2/5	1/5	0	X	X	
x_2	0	9/10	0	1	1/10	-3/10	0	X	X	
s_3	0	37/10	0	0	3/10	1/10	1	X	X	
	$Z^* = 0$		0	0	0	0	0	X	X	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

		$C_j \rightarrow$	5	8	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
x_1	5	2/5	1	0	-2/5	$\boxed{1/5}$	0	2→
x_2	8	9/10	0	1	1/10	-3/10	0	-
s_3	0	37/10	0	0	3/10	1/10	1	37
	$Z = 46/5$		0	0	-6/5	↑ -7/5	0	$\leftarrow \Delta_j$
s_2	0	2	5	0	-2	1	0	-
x_2	8	3/2	3/2	1	-1/2	0	0	-
s_3	0	7/2	-1/2	0	$\boxed{1/2}$	0	1	7→
	$Z = 12$		7	0	↑ -4	0	0	$\leftarrow \Delta_j$
s_2	0	16	3	0	0	1	2	
x_2	8	5	1	1	0	0	1/2	
s_1	0	7	-1	0	1	0	2	
	$Z = 40$		3	0	0	0	4	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z = 40$, $x_1 = 0$, $x_2 = 5$