Lecture 8

Linear programming :
Special cases in Simplex Metho

8.1 <u>Degeneracy</u>

The concept of obtaining a degenerate basic feasible solution in a LPP is known as degeneracy. The degeneracy in a LPP may arise

- At the initial stage when at least one basic variable is zero in the initial basic feasible solution.
- At any subsequent iteration when more than one basic variable is eligible to leave the basic and hence one or more variables becoming zero in the next iteration and the problem is said to degenerate. There is no assurance that the value of the objective function will improve, since the new solutions may remain degenerate. As a result, it is possible to repeat the same sequence of simplex iterations endlessly without improving the solutions. This concept is known as cycling or circling.

Rules to avoid cycling

- Divide each element in the tied rows by the positive coefficients of the key column in that row.
- Compare the resulting ratios, column by column, first in the identity and then in the body, from left to right.
- The row which first contains the smallest algebraic ratio contains the leaving variable.

Example 1

Max $Z = 3x_1 + 9x_2$
Subject to
$x_1 + 4x_2 \le 8$
$x_1 + 2x_2 \le 4$
and $x_1 \ge 0, x_2 \ge 0$

Solution

Standard LPP
Max $Z = 3x_1 + 9x_2 + 0s_1 + 0s_2$
Subject to
$x_1 + 4x_2 + s_1 = 8$
$x_1 + 2x_2 + s_2 = 4$
$x_1, x_2, s_1, s_2 \ge 0$

		$C_j \rightarrow$	3	9	0	0		
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	X_B / X_K	S_1 / X_2
s_1	0	8	1	4	1	0	2 }	1/4
s_2	0	4	1	[2]	0	1	2 }	0/2→
		Z = 0	-3	↑ -9	0	0		←Δ_j
s_1	0	0	-1	0	1	-1		
x_2	9	2	1/2	1	0	1/2		
		Z =18	3/2	0	0	9/2		

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 18, $x_1 = 0$, $x_2 = 2$

Note – Since a tie in minimum ratio (degeneracy), we find minimum of s_1 / x_k for these rows for which the tie exists.

Example 2

Max $Z = 2x_1 + x_2$

Subject to

$4x_1 + 3x_2 \leq 12$

$4x_1 + x_2 \leq 8$

$4x_1 - x_2 \leq 8$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to

$4x_1 + 3x_2 + s_1 = 12$

$4x_1 + x_2 + s_2 = 8$

$4x_1 - x_2 + s_3 = 8$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

		$C_j \rightarrow$	2	1	0	0	0			
Basic Varibles	C_B	X_B	X_1	X_2	S_1	S_2	S_3	X_B / X_K	S_1 / X_1	S_2 / X_1
s_1	0	12	4	3	1	0	0	12/4=3		
s_2	0	8	4	1	0	1	0	8/4=2	4/0=0	1/4
s_3	0	8	[4]	-1	0	0	1	8/4=2	4/0=0	0/4=0→
		Z = 0	↑ -2	-1	0	0	0	$\leftarrow \Delta_j$		
s_1	0	4	0	4	1	0	-1	4/4=1		
s_2	0	0	0	[2]	0	1	-1	0→		
x_1	2	2	1	-1/4	0	0	1/4	-		
		Z = 4	0	↑ -3/2	0	0	1/2	$\leftarrow \Delta_j$		
s_1	0	4	0	0	1	-2	[1]	0→		
x_2	1	0	0	1	0	1/2	-1/2	-		
x_1	2	2	1	0	0	1/8	1/8	16		
		Z = 4	0	0	0	3/4	↑ -1/4	$\leftarrow \Delta_j$		
s_3	0	4	0	0	1	-2	1			
x_2	1	2	0	1	1/2	-1/2	0			
x_1	2	3/2	1	0	-1/8	3/8	0			
		Z = 5	0	0	1/4	1/4	0	$\leftarrow \Delta_j$		

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 5, $x_1 = 3/2$, $x_2 = 2$

8.2 Non-existing Feasible Solution

The feasible region is found to be empty which indicates that the problem has no feasible solution.

Example

Max $Z = 3x_1 + 2x_2$

Subject to

$2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - Ma_1$

Subject to

$2x_1 + x_2 + s_1 = 2$

$3x_1 + 4x_2 - s_2 + a_1 = 12$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

		$C_j \rightarrow$	3	2	0	0	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	Min Ratio X_B / X_K
s_1	0	2	2	1	1	0	0	$2/1=2\rightarrow$
a_1	-M	12	3	4	0	-1	1	$12/4=3$
	Z= -12M		-3M-3	↑ -4M-2	0	M	0	$\leftarrow\Delta_j$
x_2	2	2	2	1	1	0	0	
a_1	-M	4	-5	0	-4	-1	1	
	Z= 4-4M		1+5M	0	2+4M	M	0	

$\Delta_j \geq 0$ so according to optimality condition the solution is optimal but the solution is called **pseudo optimal solution** since it does not satisfy all the constraints but satisfies the optimality condition. The artificial variable has a positive value which indicates there is no feasible solution.

8.3 Unbounded Solution

In some cases if the value of a variable is increased indefinitely, the constraints are not violated. This indicates that the feasible region is unbounded at least in one direction. Therefore, the objective function value can be increased indefinitely. This means that the problem has been poorly formulated or conceived.

In simplex method, this can be noticed if Δ_j value is negative to a variable (entering) which is notified as key column and the ratio of solution value to key column value is either negative or infinity (both are to be ignored) to all the variables. This indicates that no variable is ready to leave the basis, though a variable is ready to enter. We cannot proceed further and the solution is unbounded or not finite.

Example 1

Max $Z = 6x_1 - 2x_2$

Subject to

$2x_1 - x_2 \leq 2$

$x_1 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 6x_1 - 2x_2 + 0s_1 + 0s_2$

Subject to

$2x_1 - x_2 + s_1 = 2$

$x_1 + s_2 = 4$

$x_1, x_2, s_1, s_2 \geq 0$

		$C_j \rightarrow$	6	-2	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min Ratio X_B / X_K
s_1	0	2	[2]	-1	1	0	1→
s_2	0	4	1	0	0	1	4
	Z = 0		↑ -6	2	0	0	←Δ_j
x_1	6	1	1	-1/2	1/2	0	-
s_2	0	3	0	[1/2]	-1/2	1	6→
	Z = 6		0	↑ -1	3	0	←Δ_j
x_1	6	4	1	0	0	1	
x_2	-2	6	0	1	-1	2	
	Z = 12		0	0	2	2	←Δ_j

The optimal solution is $x_1 = 4$, $x_2 = 6$ and Z =12

In the starting table, the elements of x_2 are negative and zero. This is an indication that the feasible region is not bounded. From this we conclude the problem has unbounded feasible region but still the optimal solution is bounded.

Example 2

Max Z = $-3x_1 + 2x_2$
Subject to

$x_1 \leq 3$
$x_1 - x_2 \leq 0$
and $x_1 \geq 0, x_2 \geq 0$

Solution
Standard LPP
Max Z = $-3x_1 + 2x_2 + 0s_1 + 0s_2$
Subject to

$x_1 + s_1 = 3$
$x_1 - x_2 + s_2 = 0$
$x_1, x_2, s_1, s_2 \geq 0$

		$C_j \rightarrow$	-3	2	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min Ratio X_B / X_K
s_1	0	3	1	0	1	0	
s_2	0	0	1	-1	0	1	
				↑			
	Z = 0		3	-2	0	0	$\leftarrow \Delta_j$

Corresponding to the incoming vector (column x_2), all elements are negative or zero. So x_2 cannot enter the basis and the outgoing vector cannot be found. This is an indication that there exists unbounded solution for the given problem.

8.4 Multiple Optimal Solution

When the objective function is parallel to one of the constraints, the multiple optimal solutions may exist. After reaching optimality, if at least one of the non-basic variables possess a zero value in Δ_j, the multiple optimal solution exist.

Example
Max Z = $6x_1 + 4x_2$
Subject to

$2x_1 + 3x_2 \leq 30$
$3x_1 + 2x_2 \leq 24$
$x_1 + x_2 \geq 3$
and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 6x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 - Ma_1$

Subject to

$$2x_1 + 3x_2 + s_1 = 30$$
$$3x_1 + 2x_2 + s_2 = 24$$
$$x_1 + x_2 - s_3 + a_1 = 3$$
$$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$$

		$C_j \rightarrow$	6	4	0	0	0	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min Ratio X_B / X_K
s_1	0	30	2	3	1	0	0	0	15
s_2	0	24	3	2	0	1	0	0	8
a_1	-M	3	[1]	1	0	0	-1	1	3→
			↑						
	Z = -3M		-M-6	-M-4	0	0	M	0	←Δ_j
s_1	0	24	0	1	1	0	2	X	12
s_2	0	15	0	-1	0	1	[3]	X	5→
x_1	6	3	1	1	0	0	-1	X	-
							↑		
	Z = 18		0	2	0	0	-6	X	←Δ_j
s_1	0	14	0	[5/3]	1	-2/3	0	X	42/5→
s_3	0	5	0	-1/3	0	1/3	1	X	-
x_1	6	8	1	2/3	0	1/3	0	X	12
				↑					
	Z = 48		0	0	0	2	0	X	←Δ_j

Since all $\Delta_j \geq 0$, optimum solution is obtained as $x_1 = 8$, $x_2 = 0$, Max Z = 48

Since Δ_2 corresponding to non-basic variable x_2 is obtained zero, this indicates that alternate solution or multiple optimal solution also exist. Therefore the solution as obtained above is not unique.

Thus we can bring x_2 into the basis in place of s_1. The new optimum simplex table is obtained as follows

		$C_j \rightarrow$	6	4	0	0	0	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min Ratio X_B / X_K
x_2	4	42/5	0	1	3/5	-2/5	0	X	
s_3	0	39/5	0	0	1/5	1/5	1	X	
x_1	6	12/5	1	0	-2/5	3/5	0	X	
	Z = 48		0	0	0	2	0	X	←Δ_j