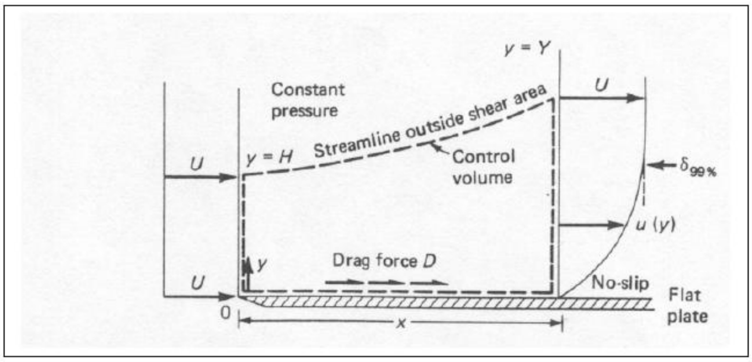
7.2 Flat-Plate Momentum Integral Analysis & Laminar approximate solution

Consider flow of a viscous fluid at high Re past a flat plate, i.e., flat plate fixed in a uniform stream of velocity $U\hat{i}$.



Boundary-layer thickness arbitrarily defined by y = $\delta_{99\%}$ (where, $\delta_{99\%}$ is the value of y at u = 0.99U). Streamlines outside $\delta_{99\%}$ will deflect an amount δ^* (**the displacement thickness**). Thus the streamlines move outward from $y = H$ at $x = 0$ to $y = Y = \delta = H + \delta^*$ at $x = x_1$.

Conservation of mass:

$$\int_{CS} \rho \underline{V} \bullet \underline{n} dA = \mathbf{0} = -\int_0^H \rho U dy + \int_0^{H+\delta^*} \rho u dy$$

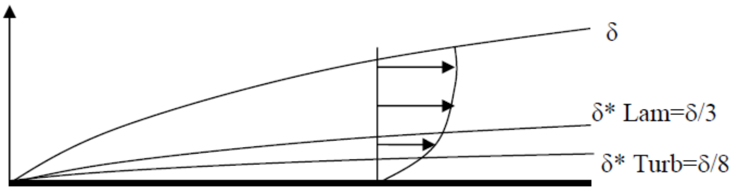
Assuming incompressible flow (constant density), this relation simplifies to

$$UH = \int_0^Y u dy = \int_0^Y (U + u - U) dy = UY + \int_0^Y (u - U) dy$$

Note: $Y = H + \delta^*$, we get the definition of displacement thickness:

$$\delta^* = \int_0^Y \left(1 - \frac{u}{U}\right) dy$$

δ^* (a function of x only) is an important measure of effect of BL on external flow. To see this more clearly, consider an alternate derivation based on an equivalent discharge/flow rate argument:



$$\underbrace{\int_{\delta^*}^{\delta} U dy}_{\text{Inviscid flow about } \delta^* \text{ body}} = \int_0^{\delta} u dy$$

Flowrate between δ^* and δ of inviscid flow=actual flowrate, i.e., inviscid flow rate about displacement body = equivalent viscous flow rate about actual body

$$\int_0^{\delta} U dy - \int_0^{\delta^*} U dy = \int_0^{\delta} u dy \Rightarrow \delta^* = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$

w/o BL - displacement effect=actual discharge

For 3D flow, in addition it must also be explicitly required that δ^* is a stream surface of the inviscid flow continued from outside of the BL.

Simple velocity profile approximations:

$$u = U(2y/\delta - y^2/\delta^2)$$

$$\left.\begin{array}{ll} u(0) = 0 & \text{no slip} \\ u(\delta) = U & \\ u_y(\delta)=0 & \end{array}\right\} \text{matching with outer flow}$$

Use velocity profile to get $C_f(\delta)$ and $\theta(\delta)$ and then integrate momentum integral equation to get $\delta(Re_x)$

$\delta^* = \delta/3$
$\theta = 2\delta/15$
$H = \delta^*/\theta = 5/2$

$$\tau_w = 2\mu U/\delta$$

$$\Rightarrow C_f = \frac{2\mu U/\delta}{1/2\rho U^2} = 2\frac{d\theta}{dx} = 2\frac{d}{dx}(2\delta/15);$$

$$\therefore \delta d\delta = \frac{15\mu dx}{\rho U}$$

$$\delta^2 = \frac{30\mu dx}{\rho U}$$

$$\left.\begin{array}{l} \delta/x = 5.5/\mathrm{Re}_x^{1/2} \\ \mathrm{Re}_x = Ux/\vartheta; \\ \delta^*/x = 1.83/\mathrm{Re}_x^{1/2} \\ \theta/x = 0.73/\mathrm{Re}_x^{1/2} \\ C_D = 1.46/\mathrm{Re}_L^{1/2} = 2C_f(L) \end{array}\right\} \text{10\% error, cf. Blasius}$$