II-Variation of parameters

The idea of variation of parameters is look for functions $v_1(x)$ and $v_2(x)$ such that $y_p = v_1 y_1 + v_2 y_2$ satisfies $ay'' + by' + cy = R(x)$ where y_1 and y_2 are solutions of $ay'' + by' + cy = 0$.We will arbitrarily decide only for v_1 and v_2 which also satisfy $v_1' y_1 + v_2' y_2 = 0 \quad \cdots (1)$

If we substitution $y_p = v_1 y_1 + v_2 y_2$ into the differential equation, we get

$$v_1' y_1' + v_2' y_2' = R(x) \qquad \cdots (2)$$

Then we solve (1) and (2) to find v_1' and v_2' then we get v_1 and v_2.

Example 1: Solve $y'' + y = \csc x$

Solution:

$$m^2 + 1 = 0 \;\Rightarrow m = \mp i$$

$$y_h = c_1 \cos x + c_2 \sin x$$

$$v_1'(x) \cos x + v_2'(x) \sin x = 0$$

$$-v_1'(x) \sin x + v_2'(x) \cos x = \csc x$$

$$v_1'(x) = \frac{\begin{vmatrix} 0 & \sin x \\ \csc x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\csc x \sin x}{\cos^2 x + \sin^2 x} = -1$$

$$\Rightarrow v_1(x) = -x$$

$$v_2'(x) = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \csc x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \csc x}{\cos^2 x + \sin^2 x} = \cot x$$

$$\Rightarrow v_2(x) = \ln|\sin x|$$

$$y_p = v_1 y_1 + v_2 y_2 = -x \cos x + \sin x \ln|\sin x|$$

$$y = y_h + y_p = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \ln|\sin x|$$

Example 2: Solve $y'' - 5y' + 6y = x^2 e^{3x}$

Solution

$$m^2 - 5m + 6 = 0 \Rightarrow (m-3)(m-2) = 0 \quad \Rightarrow \quad m_1 = 3 \ and \ m_2 = 2$$

$$y_h = c_1 e^{3x} + c_2 e^{2x}$$

$$v_1'(x) e^{3x} + v_2'(x) e^{2x} = 0$$

$$3v_1'(x) e^{3x} + 2v_2'(x) e^{2x} = x^2 e^{3x}$$

$$v_1'(x) = \frac{\begin{vmatrix} 0 & e^{2x} \\ x^2 e^{3x} & 2e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{3x} & e^{2x} \\ 3e^{3x} & 2e^{2x} \end{vmatrix}} = \frac{-x^2 e^{3x} \times e^{2x}}{e^{3x} \times 2e^{2x} - 3e^{3x} \times e^{2x}} = x^2$$

$$\Rightarrow v_1(x) = x^3/3$$

$$v_2'(x) = \frac{\begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & x^2 e^{3x} \end{vmatrix}}{\begin{vmatrix} e^{3x} & e^{2x} \\ 3e^{3x} & 2e^{2x} \end{vmatrix}} = \frac{e^{3x} \times x^2 e^{3x}}{e^{3x} \times 2e^{2x} - 3e^{3x} \times e^{2x}} = \frac{x^2 e^{6x}}{-e^{5x}} = -x^2 e^x$$

$-x^2$ & D.		e^x & I.
$-x^2$	+	e^x
$-2x$	−	e^x
-2	+	e^x
0		e^x

$$v_2(x) = -x^2 e^x + 2xe^x - 2e^x = e^x(-x^2 + 2x - 2)$$

$$y_p = v_1 y_1 + v_2 y_2 = (x^3/3)e^{3x} + e^x(-x^2 + 2x - 2)e^{2x} = e^{3x}\left((x^3/3) - x^2 + 2x - 2\right)$$

$$y = y_h + y_p = c_1 e^{3x} + c_2 e^{2x} + e^{3x}\left((x^3/3) - x^2 + 2x - 2\right)$$

Exercises

Solve the differential equations

(1) $y'' + 9y = \tan 3x$

(2) $y'' + 4y = \sin 2x \sec^2 2x$

(3) $y'' + 2y' + 2y = 3e^x \sec x$

(4) $y'' - 2y' + y = 14x^{3/2} e^x$