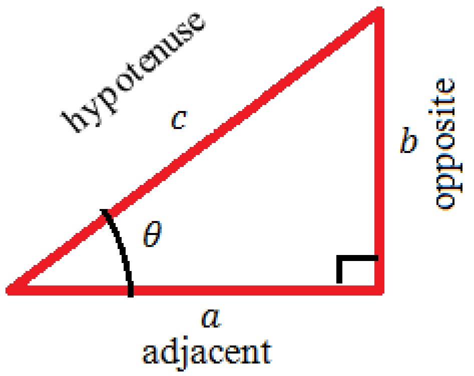
Trigonometric functions

Definitions of trigonometric functions for a right triangle

A right triangle is a triangle with a right angle (90°)



For every angle θ in the triangle, there is the side of the triangle adjacent to it, the side opposite of it and the hypotenuse such that $a^2 + b^2 = c^2$.

For angle θ, the trigonometric functions are defined as follows:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{c} \quad , \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{a}{c}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{opp}}{\text{adj}} = \frac{b}{a} \quad , \quad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\text{adj}}{\text{opp}} = \frac{a}{b}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj}} = \frac{c}{a} \quad , \quad \csc\theta = \frac{1}{\sin\theta} = \frac{\text{hyp}}{\text{opp}} = \frac{c}{b}$$

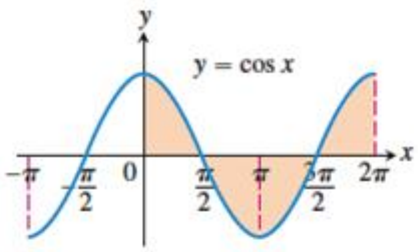
Trigonometric functions of negative angles

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$ and $\tan(-\theta) = -\tan\theta$

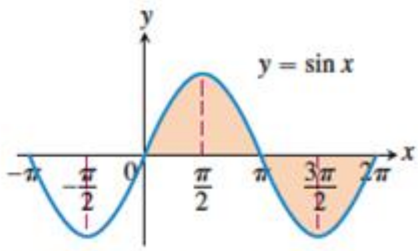
Some useful relationships among trigonometric functions

1. $\sin^2 x + \cos^2 x = 1$, $\sec^2 x - \tan^2 x = 1$, $\csc^2 x - \cot^2 x = 1$
2. $\sin 2x = 2\sin x\cos x$, $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$
3. $\sin^2 x = \dfrac{1 - \cos 2x}{2}$, $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

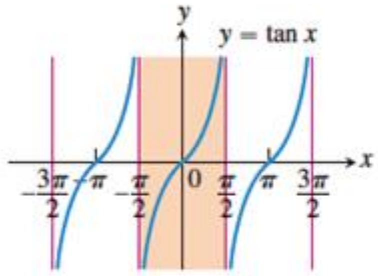
Graphs of Trigonometric Functions



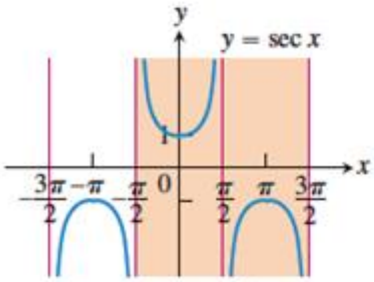
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π



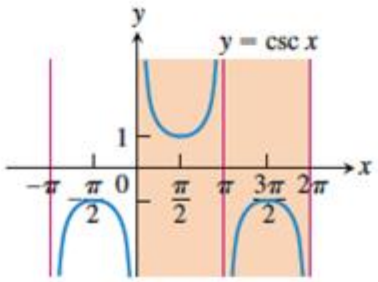
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π



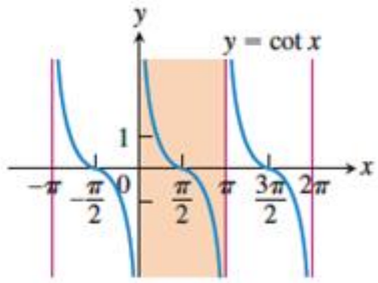
Domain: $x \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π



Domain: $x \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π



Domain: $x \ne 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π



Domain: $x \ne 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π

Derivatives of trigonometric functions

If u is a function x, the chain rule version of this differentiation rule is

1. $\frac{d}{dx}(\sin u) = \cos u \, . \frac{du}{dx}$

2. $\frac{d}{dx}(\cos u) = -\sin u \, . \frac{du}{dx}$

3. $\frac{d}{dx}(\tan u) = \sec^2 u . \frac{du}{dx}$

4. $\frac{d}{dx}(\cot u) = -\csc^2 u . \frac{du}{dx}$

5. $\frac{d}{dx}(\sec u) = \sec u \tan u \, . \frac{du}{dx}$

6. $\frac{d}{dx}(\csc u) = -\csc u \cot u \, . \frac{du}{dx}$

Example 1: Find derivatives of the functions

1. $y = \sin^2 x \quad \Rightarrow \quad y = (\sin x)^2 \quad \Rightarrow \quad \dfrac{dy}{dx} = 2\sin x \cos x = \sin 2x$

2. $y = \cos(x^2) \quad \Rightarrow \quad \dfrac{dy}{dx} = -2x \sin(x^2)$

3. $y = \tan\sqrt{x} \quad \Rightarrow \quad \dfrac{dy}{dx} = \sec^2\sqrt{x} \times \dfrac{1}{2\sqrt{x}} = \dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$

4. $y = x^2 \sec 3x \quad \Rightarrow \quad \dfrac{dy}{dx} = 3x^2 \sec 3x \tan 3x + 2x \sec 3x = x \sec 3x\,(2 + 3x \tan 3x)$

5. $y = \sqrt{\sin 2x} \quad \Rightarrow \quad y = (\sin 2x)^{1/2} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}(\sin 2x)^{-1/2} \times \cos 2x \times 2$

$$= \frac{\cos 2x}{\sqrt{\sin 2x}}$$

Example 2: If $y = \tan 2t$ and $x = \sec 2t$ show that $\dfrac{dy}{dx} = \csc 2t$

$$\frac{dy}{dt} = 2\sec^2 2t \quad , \quad \frac{dx}{dt} = 2\sec 2t \tan 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2\sec^2 2t \times \frac{1}{2\sec 2t \tan 2t} = \frac{\sec 2t}{\tan 2t}$$

$$= \frac{\frac{1}{\cos 2t}}{\frac{\sin 2t}{\cos 2t}} = \frac{1}{\sin 2t} = \csc 2t$$

Example3: If $y = \theta - \cos\theta$ and $x = \theta + \cos\theta\,;\left(0 \le \theta \le \dfrac{\pi}{2}\right)$ show that $\dfrac{dy}{dx} = (\sec\theta + \tan\theta)^2$

$$\frac{dy}{d\theta} = 1 + \sin\theta \quad \text{and} \quad \frac{dx}{d\theta} = 1 - \sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{1 + \sin\theta}{1 - \sin\theta}$$

$$\frac{dy}{dx} = \frac{1 + \sin\theta}{1 - \sin\theta} \times \frac{1 + \sin\theta}{1 + \sin\theta} = \frac{1 + 2\sin\theta + \sin^2\theta}{1 - \sin^2\theta} = \frac{1 + 2\sin\theta + \sin^2\theta}{\cos^2\theta}$$

$$\frac{dy}{dx} = \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \sec^2\theta + 2\sec\theta\tan\theta + \tan^2\theta$$

$$\therefore \quad \frac{dy}{dx} = (\sec\theta + \tan\theta)^2$$

Inverse trigonometric functions

The inverse trigonometric functions are defined to be the inverses of particular parts of the trigonometric functions; parts that do have inverses.

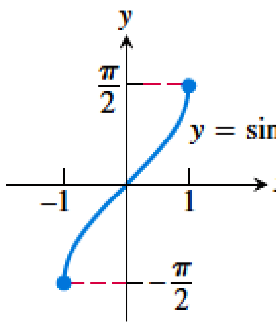
The inverse sine function, denoted by $\sin^{-1} x$ (or $\arcsin x$), is defined to be the inverse of the restricted sine function. A similar idea holds for all the other inverse trigonometric functions. It is important here to note that in this case the "-1" is not an exponent and so,

$$\sin^{-1} x \neq \frac{1}{\sin x}$$

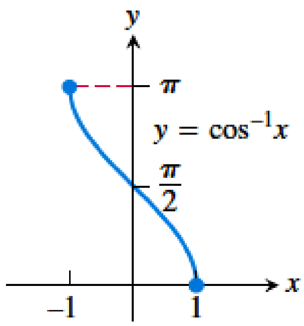
In inverse trigonometric functions the "-1" looks like an exponent but it isn't, it is simply a notation that we use to denote the fact that we're dealing with an inverse trigonometric function. It is a notation that we use in this case to denote inverse trigonometric functions. If we had really wanted exponentiation to denote 1 over sine we would use the following.

$$(\sin x)^{-1} = \frac{1}{\sin x}$$

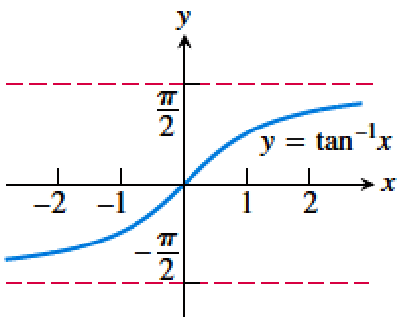
Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



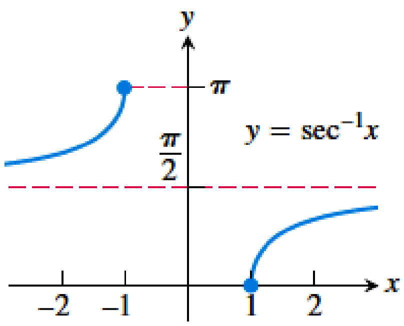
Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$



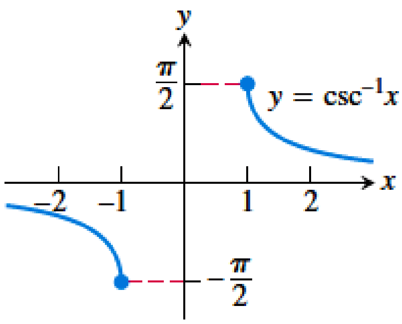
Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



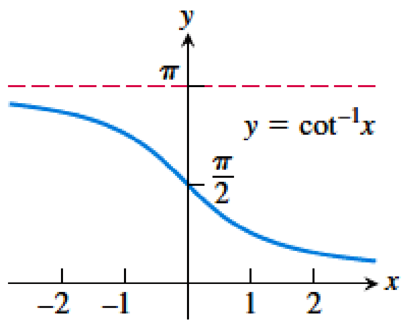
Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi, y \neq \frac{\pi}{2}$



Domain: $x \le -1$ or $x \ge 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \neq 0$



Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



Derivatives of inverse trigonometric functions

Let u be a function x, the derivatives of inverse trigonometric functions are:

1. $\dfrac{d}{dx}(\sin^{-1} u) = \dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$

2. $\dfrac{d}{dx}(\cos^{-1} u) = \dfrac{-1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$

3. $\dfrac{d}{dx}(\tan^{-1} u) = \dfrac{1}{1+u^2} \cdot \dfrac{du}{dx}$

4. $\dfrac{d}{dx}(\cot^{-1} u) = \dfrac{-1}{1+u^2} \cdot \dfrac{du}{dx}$

5. $\dfrac{d}{dx}(\sec^{-1} u) = \dfrac{1}{|u|\sqrt{u^2-1}} \cdot \dfrac{du}{dx}$

6. $\dfrac{d}{dx}(\csc^{-1} u) = \dfrac{-1}{|u|\sqrt{u^2-1}} \cdot \dfrac{du}{dx}$

Example 4: Find the derivative for

1. $y = \sin^{-1} 2x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-(2x)^2}} \times 2 = \dfrac{2}{\sqrt{1-4x^2}}$

2. $y = 3x\cos^{-1} 3x - \sqrt{1-9x^2}$

$$\frac{dy}{dx} = 3x \times \frac{-1}{\sqrt{1-(3x)^2}} \times 3 + 3\cos^{-1} 3x - \frac{-18x}{2\sqrt{1-9x^2}}$$

$$= \frac{-9x}{\sqrt{1-9x^2}} + 3\cos^{-1} 3x + \frac{9x}{\sqrt{1-9x^2}} = 3\cos^{-1} 3x$$

3. $y = 2\sqrt{x}\tan^{-1}\sqrt{x}$

$$\frac{dy}{dx} = 2\sqrt{x} \times \frac{1}{1+(\sqrt{x})^2} \times \frac{1}{2\sqrt{x}} + 2\tan^{-1}\sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{1}{1+x} + \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}$$

Exercises

Find derivative in each of the following problems $(1-4)$

1. $y = \sec^2 2x$
2. $y = x^2 \sin x + 2x\cos x - 2\sin x$
3. $y = \sqrt{x^2-1} - \sec^{-1} x$
4. $y = 2x\cos^{-1}\sqrt{x} + \sin^{-1}\sqrt{x} - 2\sqrt{x-x^2}$
5. If $y = 1-\sin\theta$ and $x = \theta - \sin\theta$ find $\dfrac{dy}{dx}$
6. If $y = \sec^{-1} t$ and $x = \sqrt{t^2-1}$ find $\dfrac{dy}{dx}$