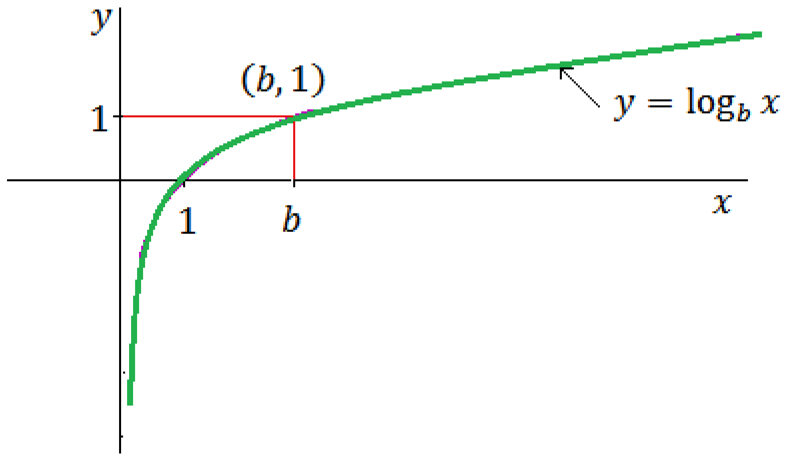
Logarithm function

The logarithm function with base b is the function $y = \log_b x$ where $b > 0$ and

The function is defined for all $x > 0$. $b \neq 1$.

Here is its graph for any base b.



Note the following:

1. For any base, the x-intercept is 1. ⇨ $\log_b 1 = 0$.

2. The graph passes through the point $(b, 1)$. ⇨ $\log_b b = 1$.

3. The graph is below the x-axis -- the logarithm is negative -- for $0 < x < 1$

4. The function is defined only for positive values of x.

5. The range of the function is all real numbers.

6. The negative y-axis is a vertical asymptote.

7. $\log_b(xy) = \log_b x + \log_b y$.

8. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$.

9. $\log_b\left(\frac{1}{x}\right) = -\log_b x$.

10. $\log_b x^y = y \log_b x$.

11. For each strictly positive real number a and b, different from 1, we have

$$\log_b x = \frac{\log_a x}{\log_a b}$$

The natural logarithm $y = \ln x$

The system of natural logarithms has the number called e as its base. e is an irrational number; its decimal value is approximately 2.71828182845904. To indicate the natural logarithm of a number we write "ln." $\ln x$ means $\log_e x$.

So we have

1. $\ln e = 1$
2. $\log_b x = \dfrac{\ln x}{\ln b}$
3. $\ln(xy) = \ln x + \ln y$
4. $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$
5. $\ln x^n = n \ln x$

Derivative of natural logarithm function

If u is a function x, then

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$$

Example 1: Find derivatives of the functions

1. $y = \ln(5x+1)$

$$\frac{dy}{dx} = \frac{1}{5x+1} \times 5 = \frac{5}{5x+1}$$

2. $y = 2x \tan^{-1} x - \ln(x^2+1)$

$$\frac{dy}{dx} = \frac{2x}{1+x^2} + 2\tan^{-1} x - \frac{2x}{x^2+1} = 2\tan^{-1} x$$

3. $y = \ln(\sin 3x)$

$$\frac{dy}{dx} = \frac{1}{\sin 3x} \times 3\cos 3x = 3\cot 3x$$

4. $y = \ln(x^2+3)^{(x^2+3)} \Rightarrow y = (x^2+3)\ln(x^2+3)$

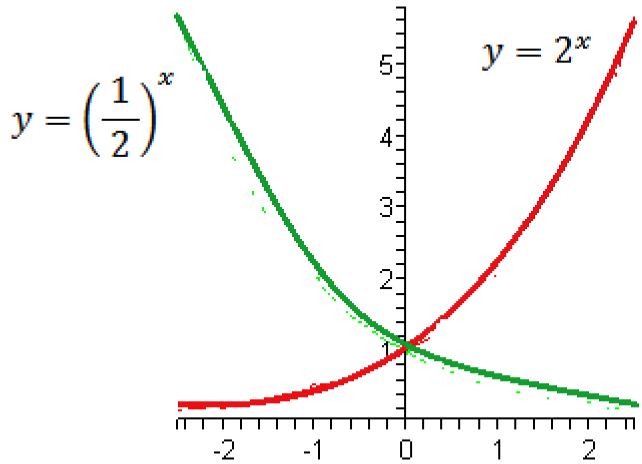
$$\frac{dy}{dx} = (x^2+3) \times \frac{2x}{(x^2+3)} + 2x\ln(x^2+3) = 2x + 2x\ln(x^2+3)$$

Exponential functions

For any positive number $a > 0$, $a \neq 1$, there is a function called an exponential function that is defined as $f(x) = a^x$

For example

$$y = 2^x \ , \ y = \left(\frac{1}{2}\right)^x$$



Now, let's talk about some of the properties of exponential functions.

1. The graph of $f(x) = a^x$ will always contain the point$(0,1)$. Or put another way, $a^0 = 1$ regardless of the value of a.
2. For every possible a, $a^x > 0$. Note that this implies that $a^x \neq 0$.
3. If $0 < a < 1$ then the graph of a^x will decrease as we move from left to right.
4. If $a > 1$ then the graph of a^x will increase as we move from left to right.
5. If $a^x = b^x$ then $a = b$.

Basic rules for exponents

1. The product rule $\quad a^x . a^y = a^{x+y}$

2. The quotient rule $\quad \dfrac{a^x}{a^y} = a^{x-y}$

3. The rule for power of a power $\quad (a^x)^y = a^{x.y}$

Natural exponential function

The function $f(x) = e^x$ is often called exponential function or natural exponential function which is an important function. The exponential function $f(x) = e^x$ is the inverse of the logarithm function $f(x) = \ln x$.

Derivatives of exponential function

If u is a function x, then

$$\frac{d}{dx}(e^u) = e^u . \frac{du}{dx}$$

Example 2: Find y' and y'' of the functions

1. $y = e^{3x-2}$

$$y' = e^{3x-2} \times 3 = 3e^{3x-2}$$

$$y'' = 3e^{3x-2} \times 3 = 9e^{3x-2}$$

2. $y = 2xe^{1-5x}$

$$y' = 2xe^{1-5x} \times (-5) + 2e^{1-5x} = e^{1-5x}(-10x+2)$$

$$y'' = e^{1-5x} \times (-10) + (-10x+2)e^{1-5x} \times (-5)$$

$$= -5e^{1-5x}(2 - 10x + 2) = 10e^{1-5x}(5x-2)$$

3. $y = e^{-3x} \sin 2x$

$$y' = 2e^{-3x} \cos 2x - 3e^{-3x} \sin 2x = e^{-3x}(\cos 2x - 3 \sin 2x)$$

$$y'' = e^{-3x}(-4 \sin 2x - 6 \cos 2x) - 3e^{-3x}(\cos 2x - 3 \sin 2x)$$

$$y'' = (5 \sin 2x - 2 \cos 2x)e^{-3x}$$

4. $y = e^{\sqrt{1-2x}}$

$$y' = \frac{-e^{\sqrt{1-2x}}}{\sqrt{1-2x}}$$

$$y'' = \frac{\sqrt{1-2x} \times \frac{e^{\sqrt{1-2x}}}{\sqrt{1-2x}} - \left(-e^{\sqrt{1-2x}}\right) \times \frac{-1}{\sqrt{1-2x}}}{1-2x} = \frac{e^{\sqrt{1-2x}} - \frac{e^{\sqrt{1-2x}}}{\sqrt{1-2x}}}{1-2x}$$

$$= \frac{e^{\sqrt{1-2x}}\left(1 - \frac{1}{\sqrt{1-2x}}\right)}{1-2x} = \frac{e^{\sqrt{1-2x}}\left(\frac{\sqrt{1-2x}-1}{\sqrt{1-2x}}\right)}{1-2x} = \frac{e^{\sqrt{1-2x}}(\sqrt{1-2x}-1)}{(1-2x)^{3/2}}$$

Solving exponential and logarithm equations

Logarithms are the "opposite" of exponentials. In practical terms, I have found it useful to think of logarithm in terms of the relationship:

$$y = \log_b x \quad \Leftrightarrow \quad x = b^y$$

$$y = \ln x \quad \Leftrightarrow \quad x = e^y$$

Physical application:

i- Radioactive Decay: The amount of a radioactive element A at time t is given by:

$$A = A_0 e^{kt}$$

Where A_0 is the initial amount of the element and k is the constant of proportionality.

Example 3: The radioactive element radium-226 has a half-life of 1620 years. If a sample initially contains 120 gm, find the constant k.

$$t = 1620 \quad \Rightarrow \quad A = \frac{1}{2}A_0 \quad \Rightarrow \quad A = \frac{1}{2} \times 120 = 60 \text{ gm}$$

$$A = A_0 e^{kt} \quad \Rightarrow \quad 60 = 120\, e^{k \times 1620}$$

$$e^{1620k} = \frac{60}{120} = 0.5$$

$$1620k = \ln(0.5)$$

$$k = \frac{\ln(0.5)}{1620} = \frac{-0.6931}{1620} = -4.28 \times 10^{-4}$$

Example 4: The radioactive element Iodine-131 has a half-life of 8 days. If a sample initially contains 5 gm . Find a function which gives the amount at any time t.

$$t = 8 \quad \Rightarrow \quad A = \frac{1}{2}A_0 \quad \Rightarrow \quad A = \frac{1}{2} \times 5 = 2.5 \text{ gm}$$

$$A = A_0 e^{kt} \quad \Rightarrow \quad 2.5 = 5\, e^{k \times 8}$$

$$e^{8k} = 0.5 \quad \Rightarrow \quad k = \frac{\ln(0.5)}{8} = -0.0866$$

$$\text{So} \quad A = A_0 e^{-0.0866\, t}$$

ii- Newton's low of cooling

The temperature T of an object at time t is given by: $T = T_s + ce^{-kt}$

Where T_s the temperature of the surrounding medium and c, k are constants.

Example 5: Placed a metal bar, at a temperature of 100°F in a room with constant temperature of 0°F. After 20 minutes the temperature of the bar is 50°F. Find c and k.

Solution: $T_s = 0°\mathrm{F}$

$$T = T_s + ce^{-kt} \;\&\; (T_s = 0) \;\Rightarrow\; T = ce^{-kt}$$

$(Initial\ condition)$ $\quad t = 0 \;,\quad T = 100°\mathrm{F} \;\Rightarrow\; 100 = ce^{-k\times 0} \;\Rightarrow\; c = 100$

$$T = 100e^{-kt}$$

To find k $\quad we\ have\ t = 20\,min \;,\; T = 50°\mathrm{F} \;\Rightarrow\; 50 = 100e^{-k\times 20}$

$$e^{-20k} = 0.5 \;\Rightarrow\; -20k = \ln(0.5)$$

$$\therefore \quad k = \frac{\ln(0.5)}{-20} = 0.0347$$

Example 6: A body at an unknown temperature is placed in a room which is held at a constant temperature of 30° F. If after 10 min the temperature of the body is 0° F and after 20 min the temperature of the body is 15° F, find the unknown initial temperature.

Solution: $T_s = 30°\mathrm{F}$

$$T = T_s + ce^{-kt} \;\&\; (T_s = 30) \;\Rightarrow\; T = 30 + ce^{-kt}$$

$$t = 10 \;,\; T = 0 \;\Rightarrow 0 = 30 + ce^{-10k}$$

$$ce^{-10k} = -30 \quad \cdots(1)$$

$$t = 20 \;,\quad T = 15 \;\Rightarrow 15 = 30 + ce^{-20k}$$

$$ce^{-20k} = -15 \quad \cdots(2)$$

$$(1) \div (2) \;\Rightarrow\; e^{10k} = 2$$

$$10\;k = \ln 2 = 0.693$$

$$k = 0.0693$$

$$\text{To find } c \;:\quad ce^{-10\times 0.0693} = -30$$

$$c = -30 \times e^{0.693} = -60$$

$$T = 30 - 60e^{-0.0693t}$$

$$t = 0 \;\Rightarrow\; T = 30 - 60e^{-0.0693\times 0} \;\Rightarrow\; T = -30°\,\mathrm{F}$$

Exercises

Find derivative in each of the following problems$(1-4)$

1. $y = \ln(x^2 + x)$
2. $y = x^3 \ln(x^2 - 2x + 5)$
3. $y = e^{\sin^{-1} x}$
4. $y = x^3 e^{-5x}$
5. The radioactive element Chromium- 51, has a half-life of 27.7 days. If a sample initially contains 75 milligrams. Find a function which gives the amount at any time t.
6. A 40°F roast is cooked in a 350°F oven. After 2 hours, the temperature of the roast is 125°F.

a. Find a formula for the temperature of the roast as a function of t.

b. The roast is done when the internal temperature reaches 165°F. When will the roast be done?

7. A body at a temperature of 50° F is placed in an oven whose temperature is kept at 7
150° F. If after 10 minutes the temperature of the body is 75° F, find the time required for the body to reach a temperature of 100° F.