Integration

This chapter covers methods that can be used to compute integrals. We will begin with integration of symbolic expressions and then consider numerical integration.

The Int Command

Let *f* be a symbolic expression in MATLAB. We can derive an expression giving the indefinite integral of *f* by writing:

```
int(f)
```

The expression *f* can be entered by creating a variable or reference first or by directly passing a quote-delimited string to int. For example, we can show that:

$$\int x\,dx = \frac{1}{2}x^2$$

(leaving out the constant of integration) by writing:

```
>> int('x')
```

MATLAB returns:

```
ans =

1/2*x^2
```

MATLAB can generate integrals that are entirely symbolic. That is, instead of:

```
>> int('x^2')

ans =

1/3*x^3
```

Consider:

```
>> int('x^n')

ans =

x^(n+1)/(n+1)
```

If we don't tell it anything, int will make assumptions about what variable to integrate. For example, we can define a trig function:

```
>> g = 'sin(n*t)';
```

If we just pass this function to int, it assumes that *t* is the integration variable:

```
>> int(g)

ans =

-1/n*cos(n*t)
```

However we can call int using the syntax int(f, v) where f is the function to integrate and v is the integration variable. Using g again we could write:

```
>> syms n
>> int(g,n)

ans =

-1/t*cos(n*t)
```

For readers taking calculus, don't forget to add the constant of integration to your answer. When creating symbolic expressions, it's not always necessary to enter them in quotes—remember to use the syms command. If we type:

```
>> g = b^t;
```

MATLAB complains, saying:

```
??? Undefined function or variable 't'.
```

We can get around this in the following way. First we call syms and tell MATLAB what we want to use for symbolic variables, and then we can define our functions without enclosing them in quotes:

```
>> syms a t
>> g = a*cos(pi*t)

g =

a*cos(pi*t)


>> int(g)

ans =

a/pi*sin(pi*t)
```

EXAMPLE 8-1
What is the integral of $f(x) = b^x$? Evaluate the resulting expression for $b = 2$, $x = 4$.

SOLUTION 8-1
We start by defining our symbolic variables:

```
>> syms b x
```

Now we define the function and integrate:

```
>> f = b^x;
>> F = int(f)

F =

1/log(b)*b^x
```

We can obtain a numerical value for the expression with the given values by calling subs. To substitute numerical values for multiple symbolic variables in one command, enclose the variable list and substitution list in curly braces { }. In this case we write:

```
>> subs(F,{b,x},{2,4})

ans =

   23.0831
```

EXAMPLE 8-2
Compute $\int x^5 \cos(9x)\,dx$.

SOLUTION 8-2
Doing this integral by hand would require integration by parts and a great deal of pain. With MATLAB, we can generate the answer on a single line:

```
>> F = int(x^5*cos(x))

F =

x^5*sin(x)+5*x^4*cos(x)-20*x^3*sin(x)-60*x^2*cos(x)+120*cos(x)+120*x*sin(x)
```

We can use the command "pretty" to have MATLAB display the answer in a more pleasing format:

```
>> pretty(F)
```

$$x^5 \sin(x) + 5x^4 \cos(x) - 20x^2 - 60x \cos(x) + 120 \cos(x) + 120x \sin(x)$$

EXAMPLE 8-3
Find $\int 3y \sec(x)\,dy$.

SOLUTION 8-3
The integrand contains two variables, so we tell MATLAB that we want to integrate with respect to *y*:

```
>> int('3*y^2*sec(x)',y)

ans =

y^3*sec(x)
```

Had we wanted to integrate with respect to x instead we would have written:

```
>> int('3*y^2*sec(x)',x)

ans =

3*y^2*log(sec(x)+tan(x))
```

Definite Integration

The int command can be used for definite integration by passing the limits over which you want to calculate the integral. If we enter int(f, a, b) then MATLAB integrates over the default independent variable and returns:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

For example:

$$\int_2^3 x\,dx = \frac{1}{2}x^2\Big|_2^3 = \frac{1}{2}(9-4) = \frac{5}{2}$$

would be calculated in MATLAB by writing:

```
>> int('x',2,3)

ans =

5/2
```

Equivalently, if we wanted MATLAB to generate the intermediate expression $\frac{1}{2}x^2$, we could write:

```
>> F = int('x')

F =

1/2*x^2
```

```
>> a = subs(F,x,3)-subs(F,x,2)

a =

    2.5000
```

EXAMPLE 8-4

What is the area under the curve $f(x) = x^2 \cos x$ for $-6 \le x \le 6$?

SOLUTION 8-4

The curve is shown in Figure 8-1. To find the area under the curve, we compute:

$$\int_{-6}^{6} x^2 \cos x \, dx$$

Let's define the function:

```
>> f = x^2*cos(x);
```

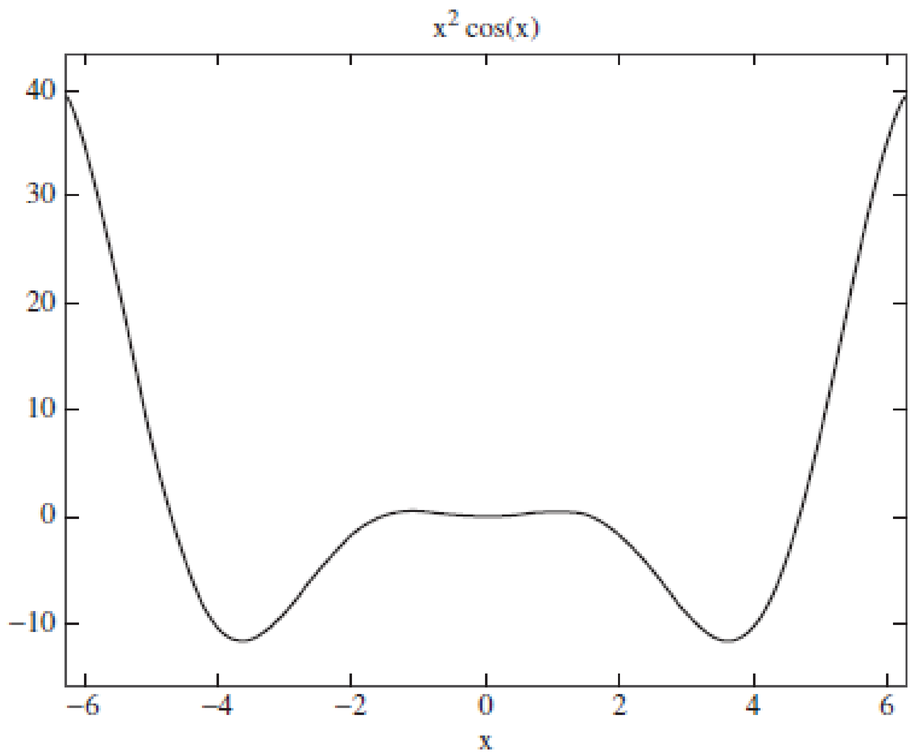

Figure 8-1 A plot of $f(x) = x^2 \cos x$ for $-6 \le x \le 6$

Now we integrate:

```
>> a = int(f,-6,6)

a =

68*sin(6)+24*cos(6)
```

To get a numerical result, we cast it with double:

```
>> double(a)

ans =

    4.0438
```

EXAMPLE 8-5
Calculate $\int_0^\infty e^{-x^2} \sin x \, dx$.

SOLUTION 8-5
We tell MATLAB that we want to evaluate the result at infinity by using inf as the upper limit:

```
>> a = int(exp(-x^2)*sin(x),0,inf)

a =

-1/2*i*pi^(1/2)*erf(1/2*i)*exp(-1/4)
```

Now we numerically evaluate the result:

```
>> double(a)

ans =

    0.4244
```

EXAMPLE 8-6
Find the volume of the solid of revolution obtained by rotating the curve e^{-x} about the x axis where $1 \le x \le 2$.

SOLUTION 8-6

The curve is shown in Figure 8-2. The volume of a solid generated by rotating a curve $f(x)$ about the x axis is given by:

$$\int_a^b \pi[f(x)]^2 dx$$

The integrand in this case is:

$$\pi(e^{-x})^2 = \pi e^{-2x}$$

The volume of the solid is then:

```
>> int(pi*exp(-2*x),1,2)

ans =

-1/2*pi*exp(-4)+1/2*pi*exp(-2)
```

Numerically we find this value is 0.1838.

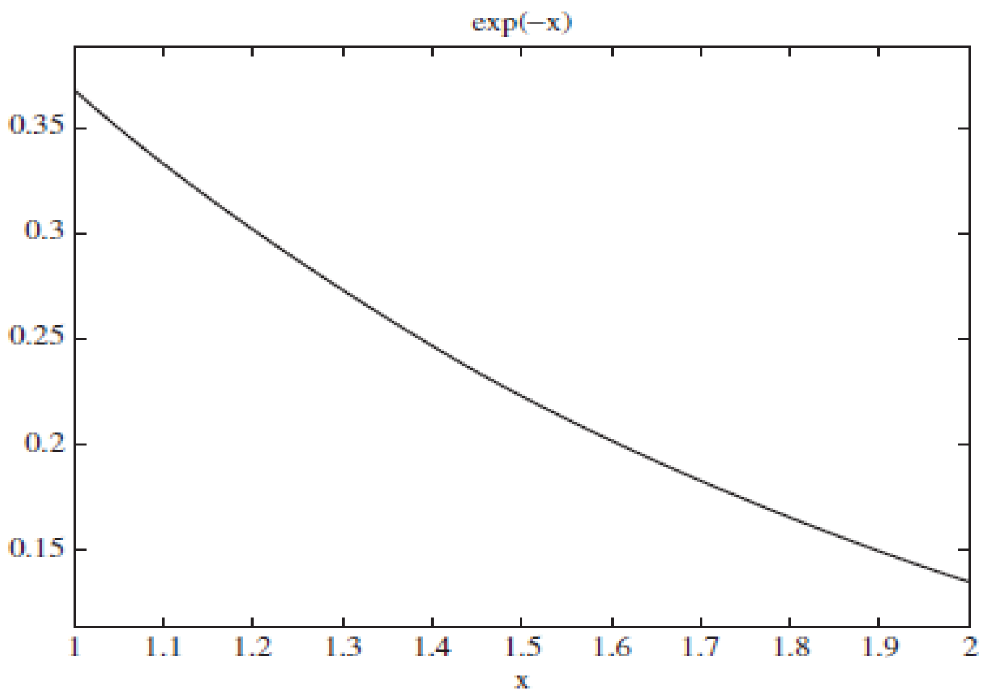

Figure 8-2 In Example 8-6 we find the volume of the solid of revolution generated by rotating the curve e^{-x} about the x axis

EXAMPLE 8-7

The sinc function, which is useful in signal processing for band-limited signals, is defined as:

$$f(x) = \frac{\sin(x)}{x}$$

Find the integral of the sinc function and sinc squared over the ranges $-20 \le x \le 20$, $-\infty < x < \infty$.

SOLUTION 8-7

Let's define both functions:

```
>> sinc = sin(x)/x;

>> sinc_squared = sinc^2;
```

The sinc-squared function has application for the description of the intensity of a light beam that has passed through a circular or square lens. First let's plot both functions, the sinc function first:

```
>> ezplot(sinc,[-20 20])
>> axis([-20 20 -0.5 1.2])
```

A plot of the sinc function is shown in Figure 8-3.
Now let's plot its square:

```
>> ezplot(sinc_squared,[-10 10])
>> axis([-10 10 -0.1 1.2])
```

A plot of the sinc-squared function is shown in Figure 8-4.
Now let's calculate the integrals for $-20 \le x \le 20$. For the sinc function we find:

```
>> a = int(sinc,-20,20)

a =

2*sinint(20)
```

We can cast this as a double to get a numerical result:

```
> double(a)

ans =

    3.0965
```

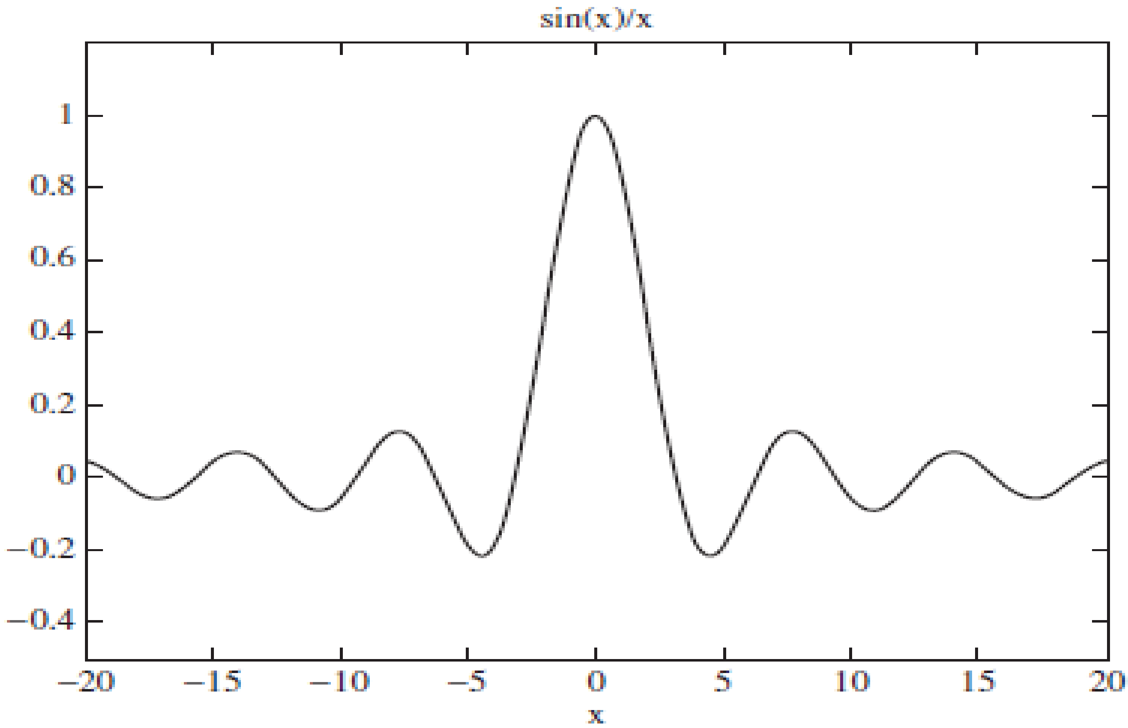

Figure 8-3 A plot of the sinc function

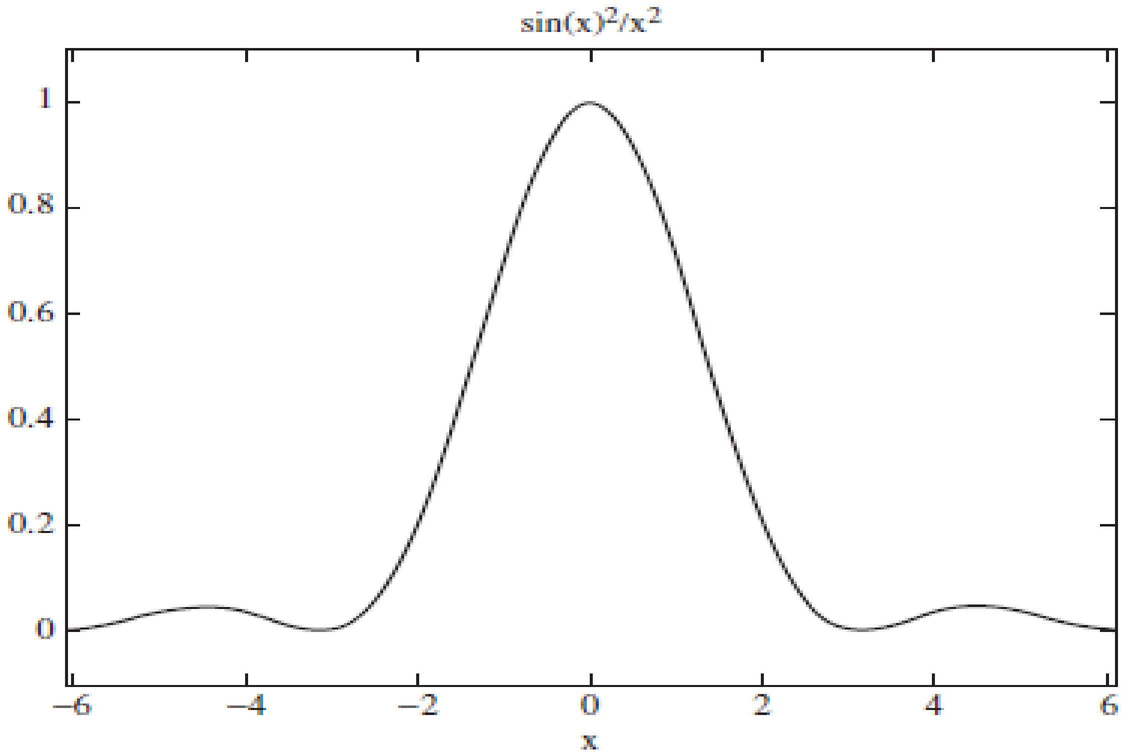

Figure 8-4 The sinc-squared function

Now we integrate sinc squared:

```
>> b = int(sinc_squared,-20,20)

b =

-1/20+1/20*cos(40)+2*sinint(40)
```

The numerical result is:

```
>> double(b)

ans =

    3.0906
```

Both results are very close. The integrals for $-\infty < x < \infty$ are:

```
>> a = int(sinc,-inf,inf)

a =

pi

>> b = int(sinc_squared,-inf,inf)

b =

pi
```

In fact what we find that as a gets bigger integrating over $-a \le x \le a$ both functions approach π. Sinc squared does so a little bit faster because the side lobes of the sinc function alternate between positive and negative and cancel out each other a little bit:

```
>> a = double(int(sinc,-50,50))

a =

    3.1032

>> b = double(int(sinc_squared,-50,50))

b =

    3.1217
```

Over the entire real line, this effect washes out and the functions both cover the same area.

Multidimensional Integration

We can compute multidimensional integrals in MATLAB by using nested int statements. Suppose that we wanted to compute the indefinite integral:

$$\int\int\int x y^2 z^5 dx\, dy\, dz$$

This can be done with:

```
>> syms x y z
>> int(int(int(x*y^2*z^5,x),y),z)

ans =

1/36*x^2*y^3*z^6
```

Definite integration proceeds analogously. We can calculate:

$$\int_1^2 \int_2^4 x^2 y\, dx\, dy$$

With the commands:

```
>> f = x^2*y;
>> int(int(f,x,2,4),y,1,2)

ans =

28
```

When computing multidimensional integrals in cylindrical and spherical coordinates, be sure to enter the correct area and volume elements—MATLAB will not do this automatically.

EXAMPLE 8-8

Find the volume of a cylinder of height h and radius a. What is the volume of a cylinder with radius $a = 3.5$ inches and height $h = 5$ inches?

SOLUTION 8-8

We will integrate using cylindrical coordinates with:

$$0 \le r \le a, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le h$$

The volume element in cylindrical coordinates is:

$$dV = r\,dr\,d\theta\,dz$$

So the volume of a cylinder of height *h* and radius *a* is:

$$V = \int_0^h \int_0^{2\pi} \int_0^a r\,dr\,d\theta\,dz$$

The commands to implement this in MATLAB are:

```
>> syms r theta z h a

>> V = int(int(int(r,r,0,a),theta,0,2*pi),z,0,h)

V =

a^2*pi*h
```

The volume of a cylinder with radius $a = 3.5$ inches and height $h = 5$ inches is:

```
>> subs(V,{a,h},{3.5,5})

ans =

  192.4226
```

The answer is expressed in cubic inches.