$\boldsymbol{cauchy - Riemann\ Equations\ \ in\ polar\ coordinates}: -$

$suppose\ f(z) = f\left(re^{i\theta}\right) = u(r,\theta) + iv(r,\theta)$

$where\ z = re^{i\theta}$

$suppose\ f\ is\ differentiable\ at\ z = z(r,\theta) we\ have.$

$f\left(re^{i\theta}\right) = u(r,\theta) + iv(r,\theta)$

$then\ differentia$ting w.r.to r we get

$e^{i\theta}\acute{f}\left(re^{i\theta}\right) = u_r + iv_r \ldots (1)$

$differentiating\ it\ w.r.to\ \theta\ we\ have$

$e^{i\theta}.ir.\acute{f}\left(re^{i\theta}\right) = u_\theta + iv_\theta \ldots (2)$

$multiplying\ (1) by\ ir\ we\ get$

$ir.e^{i\theta}.\acute{f}\left(re^{i\theta}\right) = ir\ u_r - r\ v_r$

$by\ (2) we\ get$

$ir\ u_r - r\ v_r = u_\theta + iv_\theta$

$equaling\ the\ real\ and\ imaginary\ parts\ one\ can\ have$

$ru_r = v_\theta$

$-rv_r = u_\theta$

$this\ means$

$$\left.\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ v_r &= \frac{-1}{r} u_\theta \end{aligned}\right\} \qquad (*)$$

$which\ are\ called\ the\ cauchy$
$- Riemann\ Equations\ in\ polar\ coordinates$

$, here\ we\ have$

$\acute{f}(z) = e^{-i\theta}[\ u_r + iv_r]$

$from\ (*)\ we\ get$

$$u_{rr} = \frac{-1}{r^2} v_\theta + \frac{1}{r} v_{r\theta} \ \dots (**)$$

$differentiating$

$$v_r = \frac{-1}{r} u_\theta \qquad w.r.to\ \ \theta\ \ we\ get$$

$$v_{\theta r} = \frac{-1}{r} u_{\theta\theta}$$

$but\ then$

$$u_{rr} = \frac{-1}{r^2} v_\theta + \frac{1}{r} [\frac{-1}{r} u_{\theta\theta}]$$

$$= \frac{-1}{r} . \frac{1}{r} v_\theta - \frac{1}{r^2} u_{\theta\theta}$$

$$so,\ \ u_{rr} = \frac{-1}{r} u_r - \frac{1}{r^2} u_{\theta\theta}$$

$$\Rightarrow u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$so\ u\ satisfies\ the\ polar\ form\ of\ laplacian\ equation\ similarly\ we\ can\ have.$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{i}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

$so\ of\ \ \acute{f}(z)\ \ exist\ at\ z \neq (0,0),$

$then\ the\ complement\ functions\ \ u\ and\ v\ are\ laplacian\ .$

$$example{:}-\ \ \ consider\ f(z) = \frac{1}{z}$$

$$suppose\ \ \ z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$then\ f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= \frac{1}{r} [\cos\theta - i\sin\theta]$$

$$= \frac{\cos\theta}{r} + i\,(\frac{-\sin\theta}{r})$$

$$= u(r,\theta) + iv(r,\theta)$$

where

$$u(r,\theta) = \frac{\cos\theta}{r}$$

$$v(r,\theta) = \frac{-\sin\theta}{r}$$

$$u_r = \frac{-1}{r^2}\cos\theta \quad , \quad u_\theta = \frac{-\sin\theta}{r}$$

$$v_r = \frac{1}{r^2}\sin\theta \quad , \quad v_\theta = \frac{-\cos\theta}{r}$$

then

c.r.equation are satisfied

$$\left.\begin{array}{l} u_r = \frac{1}{r} v_\theta \\ v_r = \frac{-1}{r} u_\theta \end{array}\right\}$$

$$f'(z) = e^{-i\theta}[u_r + iv_r] = e^{-i\theta}\left[\frac{-1}{r^2}\cos\theta + i\frac{\sin\theta}{r^2}\right] = \frac{-1}{r^2}e^{-i\theta}[\cos\theta - i\sin\theta]$$

$$= \frac{-1}{r^2}e^{-i\theta}.e^{-i\theta} = \frac{-1}{r^2}e^{-2i\theta} = \frac{-1}{r^2 e^{2i\theta}} = \frac{-1}{z^2}$$

Analytic function : $-$

suppose f is defined on a domain D, suppose z is a point of D, then f is said to be analytic at z if f is differentiable not only at z but also differentiable at each point of some nbhd of z.

further, f is said to be analytic on D if f is analytic at each point of D.

example: $-$

1. $f(z) = |z|^2$ *this function is differentiable only at* $(0,0)$ *, so f is no − where analytic where f is every where continuous .*

2. $f(z) = \sin x \cosh y - i\cos x \sinh y$.

here also , f is continuous every where but no − where analytic .

3. *consider* $f(z) = \frac{1}{z}$

here f is analytic in $\mathbb{C} - \{0\}$

4. $f(z) = e^x(\cos y + i\sin y)$

it has been observe that

$f'(z) = f(z)$. *so f is analytic at each point of the complex plane.*

entire function: −

A function f is said to be an entire function if f is analytic at each point of complex plane $\mathbb{C}$.

example: $-f(z) = e^x(\cos y + i\sin y)$

$$= e^z$$

note: −since the derivative of the polynomail exists every where it follows that every polynomial is an entire function.