$\boldsymbol{logarithmic\ function}$

$logarithimic\ function\ is\ defined\ at\ non - zero\ complex\ point$

$z = r.\exp(i\theta) = re^{i\theta}$

$\log(z) = \log\left(r.e^{i\theta}\right) = \log r + i\theta \qquad \left(\ as \log e^{i\theta} = i\theta \log(e) = i\theta\right)$

$$or \quad \log z = lnr + i\theta \ \ , r > 0 \ \ -\pi < \theta \leq \pi$$

$z = re^{i\theta} \ = r(cos\theta + isin\ \theta)$

$= r(\cos(\theta \pm 2k\pi) + isin(\theta \pm 2k\pi)) = re^{i(\theta \pm 2k\pi)}$
$= r\ \exp(i(\theta \pm 2k\pi))$

$\log z = \log r + i\ (\theta \pm 2k\pi) \qquad , k = 0,1,2,3, \dots$

$or\ \log z = \ln r + i(\theta \pm 2k\pi) \dots \dots (1) \quad k = 0,1,2,3, \dots$

$principls\ value\ of\ \log z$

$put\ k = 0\ \ , in\ \ \ equation(1)$

$\log z = \ln r + i\theta$

$\log z = \log|z| + i\,\text{Arg}(z)$

$now\ that \log z = \log r\ \ \pm 2k\pi i \ \ , \ \ k = 0,1,2, \dots$

$z = r.e^{i\theta}, \theta\ has\ any\ one\ of\ the\ value\ \theta = \theta + 2k\pi$

$\log z = \ln r + i\theta \dots (2)$

$that\ is \log z = \ln|z| + i \arg z\ (z \neq 0)$

$\underline{\text{properties logarithmic of complex number} : -}$

1 $-\ \exp(\log z) = z$

2 $-\ \log(e^{z}) = z + 2k\pi i \qquad , k = 0, \pm 1, \pm 2, \dots$

3 $-\ \log(1) = 2k\pi i \qquad k = 0, \pm 1, \pm 2, \dots$

4 $-\ \log(i) = \left(2k + \frac{1}{2}\right)\pi i \qquad k = 0, \pm 1, \dots$

$5 - \log e = 1 + 2k\pi i \qquad k = 0, \pm 1, \dots$

$6 - \log(-1) = (2k+1)\pi i \qquad k = 0, \pm 1, \dots$

$7 - \log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$

$8 - \log(z_1 z_2) = \log z_1 + \log z_2$

$proof$:

$let\ z_1 = r_1 \exp(i\theta_1) \qquad -\pi < \theta_1 \leq \pi\ ,\ \ r_1 > 0$

$z_2 = r_2 \exp(i\theta_2) \qquad -\pi < \theta_2 \leq \pi\ ,\ \ r_2 > 0$

now

$\log(r_1 r_2) = \log r_1 + \log r_2$

$\log z_1 = \log r_1 + i\theta_1$

$\log z_2 = \log r_2 + i\theta_2$

$\log z_1 + \log z_2 = \log r_1 + log r_2 + i(\theta_1 + \theta_2)$

$= \log(r_1 r_2) + i(\theta_1 + \theta_2) \dots \dots (1)$

$z_1 z_2 = r_1 r_2 \exp(i\theta_1) \exp(i\theta_2)$

$= r_1 r_2 \exp(i(\theta_1 + \theta_2))$

$\log z_1 z_2 = \log r_1 r_2 + i(\theta_1 + \theta_2) \qquad \dots\ \dots (2)$

$from\ (1) \& (2)\ \ we\ get$

$\log(z_1 z_2) = \log z_1 + log\ z_2$