$function: -$

$-suppose\ s\ is\ a\ set\ of\ complex\ number\ a\ function\ on\ s\ is\ a\ rule\ that\ arranges\ o$

$value\ to\ every\ member\ of\ s\ , we\ write\ it\ as\ w = f(z), z \in s.$

$-if\ u \& v\ are\ the\ real\ \&\ imaginary\ parts\ of\ w\ and\ x\ \&\ y\ are\ real\ and\ imaginary$

$part\ of\ z\ then$

$f(x+iy) = u + iv$

$u = u(x,y) \quad , \quad v = v(x,y)$

$in\ the\ polar\ form\ we\ can\ write$

$f\left(re^{i\theta}\right) = u + iv \quad , u = u(r,\theta) \quad , v = v(r,\theta)$

$in\ the\ case\ v = 0\ , then\ a\ real\ valued\ function\ on\ the\ domains.$

$example: \quad suppose\ f(z) = (\bar{z})^2 + 2z$

$if\ z = x + iy\ , then$

$f(z) = (x - iy)^2 + 2x + 2iy$

$= x^2 - y^2 - i2xy + 2x + i2y$

$= (x^2 - y^2 + 2x) + i(2y - 2xy)$

$$\boxed{\begin{aligned} u(x,y) &= (x^2 - y^2 + 2x) \\ v(x,y) &= (2y - 2xy) \end{aligned}}$$

$example: -consider\ f(z) = |z|^2$

$if\ z = x + iy\ \ then\ f(z) = |z|^2 = x^2 + y^2 = x^2 + y^2 + i0$

$so\ f\ is\ real\ value$

Chapter two concept of limit point

$suppose\ f\ is\ defined\ at\ all\ points\ in\ some\ nbhd\ of\ a\ point\ z_0\ , by\ the\ statment$

$that$

$$\lim_{z \to z_0} f(z) = w_0 \quad \ldots (*)$$

$we\ mean\ that\ w =$
$f(z)\ can\ be\ made\ arbitrary\ close\ to\ w_0\ provided\ z$

$is\ sufficiently\ close\ to\ z_0\ \ and\ distinct\ from\ it.$

$mathematically: -$

$this\ means\ for\ each\ \epsilon > 0\ arbitarly\ small\ there\ exist\ \delta > 0\ \ \ni$

$$|f(z) - w_0| < \epsilon\ if\ 0 < |z - z_0| < \delta\ note\ that\ \delta\ depends\ on\ \epsilon\ and\ z_0\ both$$

i.e. $\delta \equiv \delta(\epsilon, z_o)$.

$uniqueness\ of\ the\ limit: -$

$suppose\ \lim_{z \to z_0} f(z) = w_0\ ,\ \lim_{z \to z_0} f(z) = w_1\ ,$

$for\ \epsilon > 0\ arbitarely\ small\ there\ exist$

$\delta_1 \equiv \delta_1(\epsilon, z_0) > 0\ such\ that$

$$|f(z) - w_0| < \frac{\epsilon}{2} \quad if\ 0 < |z - z_0| < \delta_1.$$

$simillarly\ , there\ exist\ \delta_2 \equiv \delta_2(\epsilon, z_0)$

$$0 < |z - z_0| < \delta_2 \Rightarrow |f(z) - w_1| < \frac{\epsilon}{2}$$

$choose\ \delta = \min\{\delta_1, \delta_2\} > 0$

$then$

$$0 < |z - z_0| < \delta \ \Rightarrow |f(z) - w_0| < \frac{\epsilon}{2}\ \ \&\ \ |f(z) - w_1| < \frac{\epsilon}{2}$$

$but\ then$

$|w_0 - w_1| = |w_0 - f(z) + f(z) - w_1| \leq |w_0 - f(z)| + |f(z) - w_1| = |f(z) - w_0| + |f(z) - w_1| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, provider\ \ 0 < |z - z_0| < \delta$

$since\ w_0 - w_1\ is\ a fixed\ complex\ number\ \&\ \epsilon > 0\ is\ arbitarly\ small\ we\ get$

$|w_0 - w_1| = 0$

$\Rightarrow w_0 - w_1 = 0$

$\Rightarrow w_0 = w_1$

$example\ \ : -if\ f(z) = \frac{z}{\bar{z}}\ \ then$

$\lim_{z \to 0} f(z) \ldots \ldots (*) \quad does\ not\ exist\ .$

$for\ , if\ it\ did\ exist\ , it\ could\ be\ found\ by\ letting\ the\ point\ z = (x, y) approach\ the\ origin\ in\ any\ manner\ . but\ when\ z = (x, 0) is\ a\ non - zero\ point\ on\ the\ real\ axis\ .$

$$f(z) = \frac{x + i0}{x - i0} = 1$$

$and\ when\ z = (0, y) is\ a\ non - zero\ point\ on\ the\ imaginary\ axis\ ,$

$$f(z) = \frac{0 + iy}{0 - iy} = -1$$

$thus\ , by\ letting\ z$ approach the origin a long $the\ real\ axis\ we\ would\ find\ that\ the$

$desired\ limit\ is$ 1.

$an\ approach\ a\ long\ the\ imaginary\ axis\ would\ , on\ the\ other\ hand\ , yield\ the\ limit - 1\ , since\ a\ limit\ is\ unique\ , we\ must\ conclude\ that\ limit\ (*) does\ not\ exist.$

$remarks : -$

1. $\lim_{z \to z_0} f(z) = w_0 \Leftrightarrow \lim_{z \to z_0} \overline{f(z)} = \overline{w_0}$
2. $let\ c\ be\ any\ constant\ complex\ number\ then\ \lim_{z \to z_0} c = c$
3. $\lim_{z \to z_0} z^n = {z_0}^n$
4. $\lim_{z \to z_0} \bar{z}^n = \bar{z_0}^n\ (or\ \lim_{z \to z_0} \overline{z^n} = \overline{{z_0}^n})$

theorem:

suppose that

$\lim_{z \to z_0} f(z) = L \quad , \lim_{z \to z_0} g(z) = m \ , then$

1. $\lim_{z \to z_0} [f(z) \pm g(z)] = L \pm m$
2. $\lim_{z \to z_0} [f(z).\, g(z)] = L.\, m$
3. $\lim_{z \to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{L}{m} \quad if\ g(z) \neq 0$

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exerciese :-

show that

a. $\lim_{z \to 1-i} [x + i(2x + y)] = 1 + i \qquad (z = x + iy)$
b. $\lim_{z \to i} \frac{iz^3 - 1}{z + 1} = 0$
c. $\lim_{z \to z_0} (z^2 + c) = z_0^2 + c$

$$\lim_{z \to z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q(z_0)}$$